Nonlinear Fenchel conjugates and the Riemannian Difference of Convex Algorithm

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joint work with
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Nonsmooth Optimization on Riemannian Manifolds

We are looking for numerical algorithms to find

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} f(p)$$

where

- $\mathcal{M}$ is a Riemannian manifold
- $f\colon \mathcal{M} \to \overline{\mathbb{R}}$ is a function
- ⚠ f might be nonsmooth and/or nonconvex
- ⚠ $\mathcal{M}$ might be high-dimensional
- 💡 f has some “nice structure”

The Fenchel Conjugate

The Fenchel conjugate of a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is given by

$$f^*(\xi) := \sup_{x\in\mathbb{R}^n} \langle \xi, x\rangle - f(x) = \sup_{x\in\mathbb{R}^n} \begin{pmatrix} \xi \\ -1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x \\ f(x) \end{pmatrix}$$

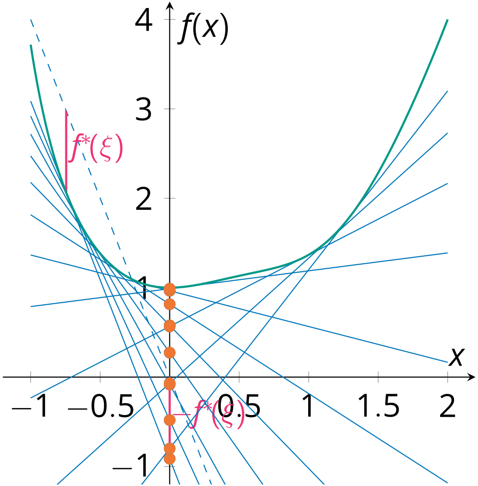
- given $\xi \in \mathbb{R}^n$: maximize the distance between $\xi^{\mathrm{T}}\cdot$ and f
- can also be written in the epigraph

The Fenchel biconjugate reads

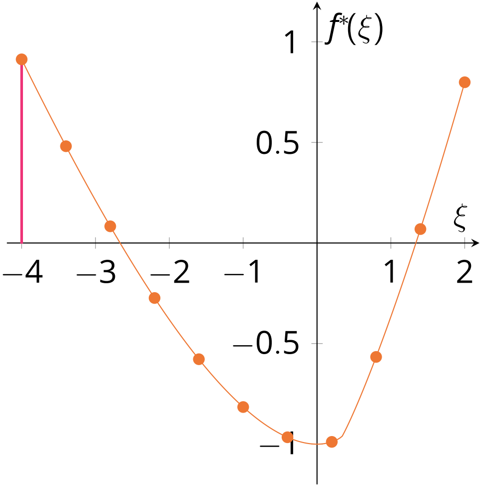
$$f^{**}(x) = (f^*)^*(x) = \sup_{\xi\in\mathbb{R}^n} \langle \xi, x\rangle - f^*(\xi).$$

Illustration of the Fenchel Conjugate

The function f



The Fenchel conjugate f^*



Applications of the Fenchel conjugate

The Fenchel conjugate is at the core of nonsmooth optimization

$$\operatorname*{arg\,min}_{x\in\mathbb{R}^n} f(x) + g(Kx)$$

as a so-called splitting method

- primal-dual (PD) algorithms [Esser, Zhang, Chan, 2010; Chambolle, Pock, 2011]
- PD with non-linear operators K [Valkonen, 2014; Mom, Langer, Sixou, 2022]
- several variants: hybrid gradient, primal/dual relaxed, linearized,...
- To derive the Difference of Convex algorithm (g concave)

Recently this has been generalised Riemannian manifolds using

- a tangent space approach [RB, Herzog, Silva Louzeiro, Tenbrinck, Vidal-Núñez, 2021]
- a tangent bundle approach [Silva Louzeiro, RB, Herzog, 2022]
- Busemann functions [de Carvalho Bento, Neto, Melo, 2023]

Formulate a framework for Fenchel conjugates on nonlinear spaces.

Nonlinear Fenchel conjugates

The Nonlinear Fenchel Conjugate

[Schiela, Herzog, RB, 2024]

In the Fenchel conjugate we use linear test functions $\varphi(x) = \langle \xi, x \rangle$.

Use arbitrary test functions

Let $\mathcal{M}$ be a set. We define the domain of the sum (difference) of two extended real-valued functions $f, g \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ as

$$\mathcal{D}(f \pm g) \coloneqq \{x \in \mathcal{M} \,|\, f(x) \pm g(x) \text{ is defined}\}.$$

Definition

The nonlinear Fenchel conjugate of $f \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ is defined as

$$\begin{aligned} f^{\circledast} \colon \mathcal{P}_{\pm\infty}(\mathcal{M}) &\to \mathbb{R}_{\pm\infty} \\ \varphi &\mapsto f^{\circledast}(\varphi) \coloneqq \sup\{\varphi(x) - f(x) \,|\, x \in \mathcal{D}(\varphi - f)\}. \end{aligned}$$

A few properties

The following properties carry over to the nonlinear case,
just being a bit careful with the domain of the test functions.

Suppose that $\mathcal{M}$ is a set and $f, g \in \mathcal{P}_{\pm\infty}(\mathcal{M})$.

[Schiela, Herzog, RB, 2024]

1. For $\alpha > 0$ and $\beta \in \mathbb{R}$,
$$\alpha f^{\circledast}(\varphi) + \beta = (\alpha f)^{\circledast}(\alpha\,\varphi + \beta) = (\alpha f - \beta)^{\circledast}(\alpha\,\varphi).$$
2. If $\mathcal{D}(f - \psi) = \mathcal{D}(\varphi + \psi) = \mathcal{M}$, then
$$(f - \psi)^{\circledast}(\varphi) = f^{\circledast}(\varphi + \psi).$$
3. If $\mathcal{D}(f + g) = \mathcal{D}(\varphi + \psi) = \mathcal{M}$ and $f^{\circledast}(\varphi) + g^{\circledast}(\psi)$ is defined, then
$$(f + g)^{\circledast}(\varphi + \psi) \leqslant f^{\circledast}(\varphi) + g^{\circledast}(\psi).$$
4. $\varphi \geqslant \psi$ and $f \leqslant g$ implies $f^{\circledast}(\varphi) \geqslant g^{\circledast}(\psi)$.
5. $f^{\circledast}$ is convex on $\mathcal{P}_{\infty}(\mathcal{M})$.

The Fenchel-Young inequality

An important inequality in the classical case is the Fenchel-Young inequality

$$f(x) + f^*(\xi) \geq \langle \xi, x \rangle$$

This carries over to the nonlinear case, with a bit of carefulness as to when the sum is defined.

Theorem (Fenchel-Young inequality)

Suppose that $f, \varphi \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ and $x \in \mathcal{M}$.
The Fenchel-Young inequalities

- $f^{\circledast}(\varphi) \geqslant \varphi(x) - f(x)$
- $f(x) \geqslant \varphi(x) - f^{\circledast}(\varphi)$
- $\varphi(x) \leqslant f(x) + f^{\circledast}(\varphi)$

hold, provided that the respective right-hand side is defined in $\mathbb{R}_{\pm\infty}$.

Nonlinear dual map

Motivation. In the classical case, we often need the adjoint K^* of K.

Definition

Suppose $\mathcal{M}$ and $\mathcal{N}$ are two non-empty sets and $A\colon \mathcal{M} \to \mathcal{N}$ is some map. The map

$$
\begin{aligned}
A^{\circledast}\colon \mathcal{P}_{\pm\infty}(\mathcal{N}) &\to \mathcal{P}_{\pm\infty}(\mathcal{M}) \\
\psi &\mapsto A^{\circledast}(\psi) := \psi \circ A
\end{aligned}
$$

is called the dual or adjoint map of A, or the pullback by A.

- $A^{\circledast}(\alpha\,\psi_1 + \psi_2) = \alpha\,A^{\circledast}(\psi_1) + A^{\circledast}(\psi_2)$ is a homomorphism
- If A is bijective, then $(f \circ A^{-1})^{\circledast} = f^{\circledast} \circ A^{\circledast}$
- more generally: defining $(f \bullet A^{-1})(y) := \inf_{x \in A^{-1}(y)} f(x)$,

  we obtain $(f \bullet A^{-1})^{\circledast} = f^{\circledast} \circ A^{\circledast}$.

Motivation: The biconjugate

- approximate f its maximal convex, lsc. minorant
- linear setting: Γ-regularization, the pointwise suppremum of continuous affine functions. [Ch. I.3 Ekeland, Temam, 1999]

$\Rightarrow$ $f^{**} \in \mathcal{P}_{\pm\infty}(V)$ coincides with Γ-regularization of f, i. e. the largest convex lsc. minorant of $f \in \mathcal{P}_{\pm\infty}(V)$

- **Fenchel-Moreau**: [Thm. 13.32 Bauschke, Combettes, 2011]

  $f \in \mathcal{P}_{\infty}(V)$ is convex, lsc. $\Leftrightarrow f^{**} = f$.

Nonlinear case.
Find a suitable subset $\mathcal{F} \subset \mathcal{P}_{\pm\infty}(\mathcal{M})$
as a generalization for affine functions.

? Can we state a biconjugation theorem as well?

$\mathcal{F}$-regularization

[Schiela, Herzog, RB, 2024]

Suppose that $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}_{\pm\infty}(\mathcal{M})$ and denote by

$$\widetilde{\mathcal{F}} := \{\varphi + c \,|\, \varphi \in \mathcal{F},\ c \in \mathbb{R}\}$$

the set of all φ that result from a shift of elements of $\mathcal{F}$.

We define the $\mathcal{F}$-regularization of $f \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ as

$$\lfloor f \rfloor_{\mathcal{F}}(x) := \sup\bigl\{\varphi(x) \,\big|\, \varphi \in \widetilde{\mathcal{F}},\ \varphi \leqslant f\bigr\}.$$

💡 $\lfloor f \rfloor_{\mathcal{F}}$ is the pointwise supremum of all minorants of f taken from $\mathcal{F}$ and its constant shifts.

In short we write: $\lfloor f \rfloor_{\mathcal{F}} = \sup\bigl\{\varphi \,\big|\, \varphi \in \widetilde{\mathcal{F}},\ \varphi \leqslant f\bigr\}$

Some properties of $\mathcal{F}$-regularization

[Schiela, Herzog, RB, 2024]

1. $f \leqslant g$ and $\mathcal{F} \subseteq \mathcal{G}$ implies $\lfloor f \rfloor_{\mathcal{F}} \leqslant \lfloor g \rfloor_{\mathcal{G}}$.
2. For $\varphi \in \mathcal{F}$ and $c \in \mathbb{R}$ we have $\lfloor f + \varphi + c \rfloor_{\mathcal{F}} = \lfloor f \rfloor_{\mathcal{F}} + \varphi + c$.
3. $\lfloor f \rfloor_{\mathcal{F}} \leqslant f$, thus $f \leqslant \lfloor f \rfloor_{\mathcal{F}} \Leftrightarrow \lfloor f \rfloor_{\mathcal{F}} = f$
4. $f \in \mathcal{F} \Rightarrow \lfloor f \rfloor_{\mathcal{F}} = f$.
5. $\mathcal{F} \subseteq \mathcal{G}$ implies $\lfloor \lfloor f \rfloor_{\mathcal{G}} \rfloor_{\mathcal{F}} = \lfloor f \rfloor_{\mathcal{F}}$.
6. if $\mathcal{F}$ is a convex cone we obtain for $\alpha_1, \alpha_2 > 0$ and $f_1, f_2 \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ with $\lfloor f_1 \rfloor_{\mathcal{F}} \not\equiv -\infty$ and $\lfloor f_2 \rfloor_{\mathcal{F}} \not\equiv -\infty$ we obtain

$$\alpha_1 \lfloor f_1 \rfloor_{\mathcal{F}} + \alpha_2 \lfloor f_2 \rfloor_{\mathcal{F}} \leqslant \lfloor \alpha_1 f_1 + \alpha_2 f_2 \rfloor_{\mathcal{F}} \leqslant \alpha_1 f_1 + \alpha_2 f_2$$

Examples

1. If $\mathcal{M}$ is a locally convex linear topological space
   - $\mathcal{F} = \mathcal{M}^*$ is its topological dual space
   - $\widetilde{\mathcal{F}}$ is the space of all continuous affine functions
   - $\lfloor f \rfloor_{\mathcal{M}^*}$ is the pointwise supremum over all affine minorants of f.
2. Suppose that $\mathcal{M}$ is a metric space.
   - Then lower semi-continuous functions $f \in \mathcal{P}_\infty(\mathcal{M})$ can be written as the pointwise supremum of continuous functions
   - For $\mathcal{F} = C(\mathcal{M})$ the set sup-cl$(\mathcal{F}) \coloneqq \left\{ \lfloor f \rfloor_{\mathcal{F}} \,\middle|\, f \in \mathcal{P}_{\pm\infty}(\mathcal{M}) \right\}$ consists of the cone of lower semi-continuous functions in $\mathcal{P}_\infty(\mathcal{M})$
3. alternative generalization: the c-conjugate [Martínez-Legaz, 2005]

   For a coupling function $c\colon \mathcal{M} \times \mathcal{N} \to \mathbb{R}_{\pm\infty}$ defined as

$$f^c(y) \coloneqq \sup_{x \in \mathcal{M}} c(x,y) - f(x) \quad \text{for } y \in \mathcal{N}.$$

   This generalizes duality pairing instead of the set of test functions.

$\mathcal{F}$-biconjugates

[Schiela, Herzog, RB, 2024]

- We denote the restriction of the conjugate $f^{\circledast} \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ to $\mathcal{F}$ by

$$f^{\circledast}|_{\mathcal{F}} \colon \mathcal{F} \to \mathbb{R}_{\pm\infty}$$

- Let the evaluation (Dirac) functions be given by

$$\delta_x \colon \mathcal{P}_{\pm\infty}(\mathcal{M}) \to \mathbb{R}_{\pm\infty}, \qquad \varphi \mapsto \delta_x(\varphi) \coloneqq \varphi(x).$$

➜ $\delta_x|_{\mathcal{F}}$, $\mathcal{F} \subset \mathcal{P}_{\pm\infty}(\mathcal{M})$ linear, is a linear function and continuous.

Definition

Suppose that $\mathcal{F}$ is a linear subspace of $\mathcal{P}(\mathcal{M})$.
We define the $\mathcal{F}$-biconjugate $f_{\mathcal{F}}^{\circledast\circledast}$ of $f \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ as

$$f_{\mathcal{F}}^{\circledast\circledast} \colon \mathcal{M} \to \mathbb{R}_{\pm\infty}, \qquad x \mapsto (f^{\circledast}|_{\mathcal{F}})^{\circledast}(\delta_x).$$

Note. We employ the embedding of $\mathcal{M}$ into the dual space of $\mathcal{F}$ via

$$J_{\mathcal{M}\to\mathcal{F}'} \colon \mathcal{M} \to \mathcal{F}', \qquad x \mapsto \delta_x.$$

$\mathcal{F}$-biconjugate theorem

Remember.
For the classical Fenchel biconjugate the set $\mathcal{F}$ are all affine functions and $\lfloor f \rfloor_{\mathcal{F}}$ is largest convex lsc. minorant of $f \in \mathcal{P}_{\pm\infty}(V)$

Theorem [Schiela, Herzog, RB, 2024]

Suppose that $\mathcal{F}$ is a linear subspace of $\mathcal{P}(\mathcal{M})$. Thn, the $\mathcal{F}$-biconjugate satisfies $f_{\mathcal{F}}^{\circledast\circledast} = \lfloor f \rfloor_{\mathcal{F}}$ for all $f \in \mathcal{P}_{\pm\infty}(\mathcal{M})$.

➜ If $f = \lfloor f \rfloor_{\mathcal{F}}$, or in other words f agrees with the pointwise supremum of all minorants from $\mathcal{F}$, then we recover f from its $\mathcal{F}$-biconjugate.

Motivation: The subdifferential

With the Fenchel conjugate $f^*\colon V^* \to \mathbb{R}_{\pm\infty}$ of a proper, convex, lsc. function $f\colon V \to \mathbb{R}_{\pm\infty}$ on a vector space V we have

$$\xi \in \partial f(x) \quad \text{if and only if} \quad x \in \partial f^*(\xi)$$

➜ we can characterize both subdifferentials.

Nonlinear case.
We need "more structure on $\mathcal{M}$" to define a subdifferential of f.

In practice/numerics we use Riemannian manifolds.

In the following we consider a manifold $\mathcal{M}$, that is locally homeomorphic to a Banach space $\mathcal{X}$, or a Banach manifold for short.

The viscosity Fréchet Subdifferential

A function $f \in \mathcal{P}_{\pm\infty}(\mathcal{M})$ is lower semi-continuous at $x \in \mathcal{M}$ if,
$\forall \varepsilon > 0 \;\; \exists$ a neighbourhood $\mathcal{U}$ of x s.t. that $f(y) \geqslant f(x) - \varepsilon$ for all $y \in \mathcal{U}$.
We denote by $\operatorname{lsc}_\infty(\mathcal{M})$ the set of all functions that are lower semi-continuous at every $x \in \mathcal{M}$.

Definition

Suppose that $\mathcal{M}$ is a C^1-Banach manifold, $f \in \operatorname{lsc}_\infty(\mathcal{M})$, $x \in \mathcal{M}$ and $f(x) \neq +\infty$.
The (viscosity) Fréchet subdifferential $\partial_F f(x)$ of f is defined as follows:

$$\partial_F f(x) := \left\{ \varphi'(x) \,\middle|\, \varphi \in C^1(\mathcal{M}),\ f - \varphi \text{ attains a local minimum at } x \right\} \subseteq \mathcal{T}_x^* \mathcal{M},$$

where $\mathcal{T}_x^*\mathcal{M} := (\mathcal{T}_x\mathcal{M})^*$ denotes the cotangent space at x.
In case $f(x) = +\infty$, we set $\partial_F f(x) := \emptyset$.

Subdifferential Classification

Theorem [Schiela, Herzog, RB, 2024]

Suppose that $\mathcal{M}$ is a C^1-Banach manifold.
Let $x \in \mathcal{M}$, f be lower semicontinuous at every $x \in \mathcal{M}$ and $\varphi \in C^1(\mathcal{M})$.

1. *If $f^{\circledast}(\varphi) = \varphi(x) - f(x)$, i. e. we have equality in the Fenchel-Young inequality,*
   then $\varphi'(x) \in \partial_F f(x)$ and the Dirac function $\delta_x \in \partial(f^{\circledast}|_{C^1(\mathcal{M})})(\varphi)$.
2. *Conversely, if $\delta_x \in \partial(f^{\circledast}|_{C^1(\mathcal{M})})(\varphi)$, then $f^{\circledast}(\varphi) = \varphi(x) - f(x)$.*

Motivation: Infimal convolution

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Infimal convolution on a vector space $\mathcal{M} = V$ is defined as

$$(f \star_{\inf} g)(x) := \inf_{y \in \mathcal{M}} \{f(y) + g(x-y)\}\,.$$

The infimal convolution formula shows that [Prop. 13.21 Bauschke, Combettes, 2011]

$$(f \star_{\inf} g)^* = f^* + g^*$$

Nonlinear case.
We need even "slightly more structure" to generalise infimal convolution, a way to define "$x - y \in \mathcal{M}$" to be precise.

? Can we then get the same result for the nonlinear Fenchel conjugate? And what is a suitable restriction of $\mathcal{P}_{\pm\infty}(\mathcal{M})$ then?

Using Lie groups

Let

- $\mathcal{M}$ be a Riemannian manifold
- $\cdot\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be a smooth group operation
- ➜ $(\mathcal{M}, \cdot)$ is a Lie group.

We generalize infimal convolution to functions $f, g \in \mathcal{P}_\infty(\mathcal{M})$ as [Bachir, 2015]

$$(f \star_{\inf} g)(x) := \inf_{y \in \mathcal{M}} f(x \cdot y^{-1}) + g(y) = \inf_{z \in \mathcal{M}} f(z) + g(z^{-1} \cdot x).$$

Consider the linear space of group homomorphisms

$$\mathcal{H} := \operatorname{Hom}((\mathcal{M}, \cdot), (\mathbb{R}, +))$$

Then we get the relation [Schiela, Herzog, RB, 2024]

$$(f \star_{\inf} g)^{\circledast}(\varphi) = f^{\circledast}(\varphi) + g^{\circledast}(\varphi) \quad \text{for all } \varphi \in \mathcal{H}.$$

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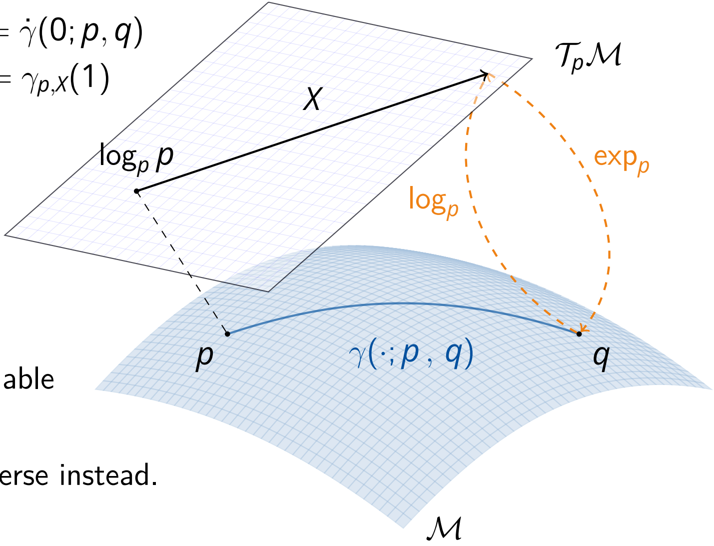
The Riemannian Difference of Convex Algorithm

A Riemannian Manifold $\mathcal{M}$

Notation.

- Logarithmic map $\log_p q = \dot\gamma(0; p, q)$
- Exponential map $\exp_p X = \gamma_{p,X}(1)$
- Geodesic $\gamma(\cdot; p, q)$
- Tangent space $\mathcal{T}_p\mathcal{M}$
- inner product $(\cdot, \cdot)_p$



Numerics.

$\exp_p$ and $\log_p$ maybe not available efficiently/ in closed form

$\Rightarrow$ use a retraction and its inverse instead.

(Geodesic) Convexity

[Sakai, 1996; Udriște, 1994]

A set $\mathcal{C} \subset \mathcal{M}$ is called (strongly geodesically) convex
if for all $p, q \in \mathcal{C}$ the geodesic $\gamma(\cdot; p, q)$ is unique and lies in $\mathcal{C}$.

A function $f\colon \mathcal{C} \to \overline{\mathbb{R}}$ is called (geodesically) convex
if for all $p, q \in \mathcal{C}$ the composition $f(\gamma(t; p, q)), t \in [0, 1]$, is convex.

The Riemannian Subdifferential

Let $\mathcal{C}$ be a convex set.

The subdifferential of f at $p \in \mathcal{C}$ is given by [Ferreira, Oliveira, 2002; Lee, 2003; Udriște, 1994]

$$\partial_{\mathcal{M}} f(p) := \bigl\{\xi \in \mathcal{T}_p^*\mathcal{M} \,\big|\, f(q) \geq f(p) + \langle \xi \,, \log_p q \rangle_p \ \text{ for } q \in \mathcal{C} \bigr\},$$

where

- $\mathcal{T}_p^*\mathcal{M}$ is the dual space of $\mathcal{T}_p\mathcal{M}$, also called cotangent space
- $\langle \cdot \,, \cdot \rangle_p$ denotes the duality pairing on $\mathcal{T}_p^*\mathcal{M} \times \mathcal{T}_p\mathcal{M}$
- numerically we use musical isomorphisms $X = \xi^\flat \in \mathcal{T}_p\mathcal{M}$ to obtain a subset of $\mathcal{T}_p\mathcal{M}$

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Difference of Convex

We aim to solve

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} f(p)$$

where

- $\mathcal{M}$ is a Riemannian manifold
- $f\colon \mathcal{M} \to \mathbb{R}$ is a difference of convex function, i. e. of the form

$$f(p) = g(p) - h(p)$$

- $g, h\colon \mathcal{M} \to \overline{\mathbb{R}}$ are convex, lower semicontinuous, and proper

The Euclidean DCA

Idea 1. At $x^{(k)}$, approximate $h(x)$ by its affine minorization

$$h_k(x) := h(x^{(k)}) + \langle x - x^{(k)}, y^{(k)} \rangle \text{ for some } y^{(k)} \in \partial h(x^k)$$

$\Rightarrow$ iteratively minimize $g(x) - h_k(x) = g(x) - h(x^{(k)}) - \langle x - x^{(k)}, y^{(k)} \rangle$

Idea 2. Using duality theory finding a new $y^{(k)} \in \partial h(x^{(k)})$ is equivalent to

$$y^{(k)} \in \operatorname*{arg\,min}_{y \in \mathbb{R}^n} \Bigl\{ h^*(y) - g^*(y^{(k-1)}) - \langle y - y^{(k-1)}, x^{(k)} \rangle \Bigr\}$$

Idea 3. Reformulate 2 using a proximal map $\Rightarrow$ DCPPA
on manifolds this was done in [Almeida, Neto, Oliveira, Souza, 2020; Souza, Oliveira, 2015]

In the Euclidean case, all three models are equivalent.

A Fenchel Duality on a Hadamard Manifold

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Let

- $T\mathcal{M} = \dot{\bigcup}_p T_p\mathcal{M}$ denote the tangent bundle
- analogously $T^*\mathcal{M}$ denotes the cotangent bundle
- $\mathcal{M}$ be a Hadamard manifold (non-positive sectional curvature).

Definition [Silva Louzeiro, RB, Herzog, 2022]

Let $f\colon \mathcal{M} \to \overline{\mathbb{R}}$.
The Fenchel conjugate of f is the function $f^*\colon \mathcal{T}^*\mathcal{M} \to \overline{\mathbb{R}}$ defined by

$$f^*(p,\xi) \coloneqq \sup_{q\in\mathcal{M}} \Bigl\{ \langle \xi, \log_p q\rangle - f(q) \Bigr\}, \qquad (p,\xi) \in \mathcal{T}^*\mathcal{M}.$$

The Dual Difference of Convex Problem

Given the Difference of Convex problem

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - h(p)$$

and the Fenchel duals g^* and h^*,
we can state the dual difference of convex problem as

[RB, Ferreira, Santos, Souza, 2024]

$$\operatorname*{arg\,min}_{(p,\xi)\in T^*\mathcal{M}} h^*(p,\xi) - g^*(p,\xi).$$

On $\mathcal{M} = \mathbb{R}^n$ this indeed simplifies to the classical dual problem.

Theorem. [RB, Ferreira, Santos, Souza, 2024]

$$\inf_{(q,X)\in\mathcal{T}^*\mathcal{M}} \Bigl\{h^*(q,X) - g^*(q,X)\Bigr\} = \inf_{p\in\mathcal{M}} \{g(p) - h(p)\}\,.$$

The Dual Difference of Convex Problem

The primal and dual Difference of Convex problem

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - h(p) \qquad \text{and} \qquad \operatorname*{arg\,min}_{(p,\xi)\in T^*\mathcal{M}} h^*(p,\xi) - g^*(p,\xi)$$

are equivalent in the following sense.

Theorem. [RB, Ferreira, Santos, Souza, 2024]

If p^* is a solution of the primal problem, then $(p^*, \xi^*) \in T^*\mathcal{M}$ is a solution for the dual problem for all $\xi^* \in \partial_{\mathcal{M}} h(p^*) \cap \partial_{\mathcal{M}} g(p^*)$.

If $(p^*, \xi^*) \in T^*\mathcal{M}$ is a solution of the dual problem for some $\xi^* \in \partial_{\mathcal{M}} h(p^*) \cap \partial_{\mathcal{M}} g(p^*)$, then p^* is a solution of the primal problem.

Derivation of the Riemannian DCA

We consider the first order Taylor approximation of h at some point $p^{(k)}$:
With $\xi \in \partial h(p^{(k)})$ we set

$$h_k(p) := h(p^{(k)}) + \langle \xi \,, \log_{p^{(k)}} p \rangle_{p^{(k)}}$$

Using musical isomorphisms we identify $X = \xi^\sharp \in T_p\mathcal{M}$,
where we call X a subgradient. Locally h_k minorizes h, i. e.

$$h_k(q) \leq h(q) \quad \text{locally around} \quad p^{(k)}$$

$\Rightarrow$ Use $-h_k(p)$ as upper bound for $-h(p)$ in $f = g - h$.

Note. On $\mathbb{R}^n$ the function h_k is linear.
On a manifold h_k is nonlinear and not even necessarily convex,
even on a Hadamard manifold.

The Riemannian DC Algorithm

[RB, Ferreira, Santos, Souza, 2024]

Input: An initial point $p^{(0)} \in \operatorname{dom}(g)$, g and $\partial_{\mathcal{M}} h$

1: Set $k = 0$.

2: **while** not converged **do**

3: Take $X^{(k)} \in \partial_{\mathcal{M}} h(p^{(k)})$

4: Compute the next iterate $p^{(k+1)}$ as

$$p^{(k+1)} \in \operatorname*{arg\,min}_{p \in \mathcal{M}} g(p) - \bigl(X^{(k)}, \log_{p^{(k)}} p\bigr)_{p^{(k)}}. \qquad (*)$$

5: Set $k \leftarrow k + 1$

6: **end while**

Note. In general the subproblem $(*)$ can not be solved in closed form. But an approximate solution yields a good candidate.

For example: Given g, $p^{(k)}$, and $X^{(k)}$ and $\operatorname{grad} g \Rightarrow$ Gradient descent.

Convergence of the Riemannian DCA

Let $\{p^{(k)}\}_{k\in\mathbb{N}}$ and $\{X^{(k)}\}_{k\in\mathbb{N}}$ be the iterates and subgradients of the RDCA.

Theorem. [RB, Ferreira, Santos, Souza, 2024]

If $\bar{p}$ is a cluster point of $\{p^{(k)}\}_{k\in\mathbb{N}}$, then $\bar{p} \in \operatorname{dom}(g)$ and there exists a cluster point $\bar{X}$ of $\{X^{(k)}\}_{k\in\mathbb{N}}$ s. t. $\bar{X} \in \partial g(\bar{p}) \cap \partial h(\bar{p})$.

$\Rightarrow$ Every cluster point of $\{p^{(k)}\}_{k\in\mathbb{N}}$, if any, is a critical point of f.

Proposition. [RB, Ferreira, Santos, Souza, 2024]

Let g be σ-strongly (geodesically) convex. Then

$$f(p^{(k+1)}) \leq f(p^{(k)}) - \frac{\sigma}{2} d^2(p^{(k)}, p^{(k+1)})$$

and $\displaystyle\sum_{k=0}^{\infty} d^2(p^{(k)}, p^{(k+1)}) < \infty$, so in particular $\displaystyle\lim_{k\to\infty} d(p^{(k)}, p^{(k+1)}) = 0$.

A Numerical Example

The Difference of Convex Algorithm in Manopt.jl

The algorithm is implemented and released in Julia using `Manopt.jl`[1].
It can be used with any manifold from `Manifolds.jl`

A solver call looks like

```
q = difference_of_convex_algorithm(M, f, g, ∂h, p0)
```

where one has to implement `f(M, p)`, `g(M, p)`, and `∂h(M, p)`.

- a sub problem is generated if keyword `grad_g=` is set
- an efficient version of its cost and gradient is provided
- you can specify the sub-solver using `sub_state=`
  to also set up the specific parameters of your favourite algorithm

[1]see https://manoptjl.org/stable/solvers/difference_of_convex/

Rosenbrock and First Order Methods

Problem. We consider the classical Rosenbrock example[2]

$$\operatorname*{arg\,min}_{x\in\mathbb{R}^2} a\bigl(x_1^2 - x_2\bigr)^2 + \bigl(x_1 - b\bigr)^2,$$

where $a, b > 0$, usually $b = 1$ and $a \gg b$, here: $a = 2 \cdot 10^5$.

Known Minimizer $x^* = \begin{pmatrix} b \\ b^2 \end{pmatrix}$ with cost $f(x^*) = 0$.

Goal. Compare first-order methods, e. g. using the (Euclidean) gradient

$$\nabla f(x) = \begin{pmatrix} 4a(x_1^2 - x_2) \\ -2a(x_1^2 - x_2) \end{pmatrix} + \begin{pmatrix} 2(x_1 - b) \\ 0 \end{pmatrix}$$

[2] available online in ManoptExamples.jl

A “Rosenbrock-Metric” on $\mathbb{R}^2$

In our Riemannian framework, we can introduce a new metric on $\mathbb{R}^2$ as

$$G_p := \begin{pmatrix} 1 + 4p_1^2 & -2p_1 \\ -2p_1 & 1 \end{pmatrix}, \text{ with inverse } G_p^{-1} = \begin{pmatrix} 1 & 2p_1 \\ 2p_1 & 1 + 4p_1^2 \end{pmatrix}.$$

We obtain $(X, Y)_p = X^{\mathrm{T}} G_p Y$

The exponential and logarithmic map are given as

$$\exp_p(X) = \begin{pmatrix} p_1 + X_1 \\ p_2 + X_2 + X_1^2 \end{pmatrix}, \qquad \log_p(q) = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 - (q_1 - p_1)^2 \end{pmatrix}.$$

`Manifolds.jl`:
Implement these functions on `MetricManifold(ℝ^2, RosenbrockMetric())`.

The Riemannian Gradient w.r.t. the new Metric

Let $f\colon \mathcal{M} \to \mathbb{R}$. Given the Euclidean gradient $\nabla f(p)$, its Riemannian gradient $\operatorname{grad} f\colon \mathcal{M} \to T\mathcal{M}$ is given by

$$\operatorname{grad} f(p) = G_p^{-1} \nabla f(p).$$

While we could implement this denoting $\nabla f(p) = \begin{pmatrix} f_1'(p) & f_2'(p) \end{pmatrix}^\mathrm{T}$ using

$$\Big\langle \operatorname{grad} f(q), \log_q p \Big\rangle_q = (p_1 - q_1) f_1'(q) + (p_2 - q_2 - (p_1 - q_1)^2) f_2'(q),$$

but it is automatically done in `Manopt.jl`.

The Experiment Setup

Algorithms. We now compare

1. The Euclidean gradient descent algorithm on $\mathbb{R}^2$,
2. The Riemannian gradient descent algorithm on $\mathcal{M}$,
3. The Difference of Convex Algorithm on $\mathbb{R}^2$,
4. The Difference of Convex Algorithm on $\mathcal{M}$.

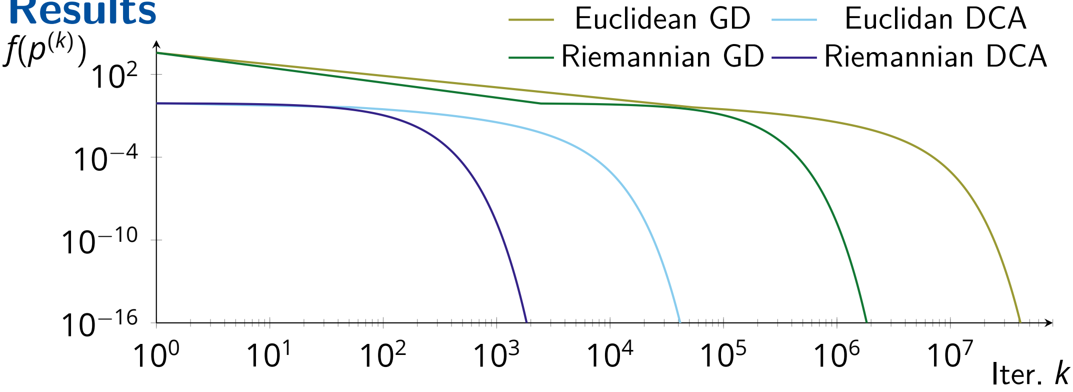
For DCA third we split f into $f(x) = g(x) - h(x)$ with

$$g(x) = a\big(x_1^2 - x_2\big)^2 + 2\big(x_1 - b\big)^2 \quad \text{and} \quad h(x) = \big(x_1 - b\big)^2.$$

Initial point. $p_0 = \frac{1}{10}\begin{pmatrix}1\\2\end{pmatrix}$ with cost $f(p_0) \approx 7220.81$.

Stopping Criterion.
$d_{\mathcal{M}}(p^{(k)}, p^{(k-1)}) < 10^{-16}$ or $\lVert \operatorname{grad} f(p^{(k)})\rVert_p < 10^{-16}$.

Results



Algorithm	**Runtime** (sec.)	**# Iterations**
Euclidean GD	305.567	53 073 227
Euclidean DCA	58.268	50 588
Riemannian GD	18.894	2 454 017
Riemannian DCA	7.704	2 459

Summary

Nonlinear Fenchel Conjugate generalises the Fenchel conjugate.
A lot of properties can be proven more generally as well:

- Fenchel-Young inequality
- Biconjugate theorem
- Subdifferential classification on manifolds
- Infimal convolution on Lie groups

The Difference of Convex Algorithm to solve a nonsmooth, nonconvex problems on manifold of the form

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - h(p)$$

- Relation to Fenchel Duality on Hadamard manifolds
- Convergence on Hadamard manifolds
- available in `Manopt.jl` for all manifolds form `Manifolds.jl`

Selected References

RB (2022). "Manopt.jl: Optimization on Manifolds in Julia". *Journal of Open Source Software* 7.70, p. 3866. DOI: 10.21105/joss.03866.

RB; O. P. Ferreira; E. M. Santos; J. C. d. O. Souza (2024). "The difference of convex algorithm on Hadamard manifolds". *Journal of Optimization Theory and Applications*. DOI: 10.1007/s10957-024-02392-8. arXiv: 2112.05250.

RB; R. Herzog; M. Silva Louzeiro; D. Tenbrinck; J. Vidal-Núñez (2021). "Fenchel duality theory and a primal-dual algorithm on Riemannian manifolds". *Foundations of Computational Mathematics* 21.6, pp. 1465–1504. DOI: 10.1007/s10208-020-09486-5. arXiv: 1908.02022.

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Schiela, A.; R. Herzog; RB (2024). *Nonlinear Fenchel conjugates*. arXiv: 2409.04492.

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Interested in Numerical Differential Geometry? Join numdiffgeo.zulipchat.com!

ronnybergmann.net/talks/2025-Saarbruecken-Nonlinear-Fenchel-Conjugates.pdf

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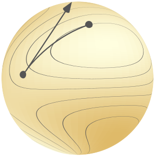
More on Manopt.jl

Manopt.jl



Goal. Provide optimization algorithms on Riemannian manifolds.

Features. Given a `Problem p` and a `SolverState s`,
implement `initialize_solver!(p, s)` and `step_solver!(p, s, i)`
$\Rightarrow$ an algorithm in the `Manopt.jl` interface

Highlevel interfaces like `gradient_descent(M, f, grad_f)`
on any manifold `M` from `Manifolds.jl`.

All provide debug output, recording, cache & counting capabilities, as well as a library of step sizes and stopping criteria.

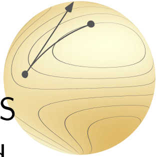
Manopt family.

manoptjl.org [RB, 2022]

manopt.org [Boumal, Mishra, Absil, Sepulchre, 2014]

pymanopt.org [Townsend, Koep, Weichwald, 2016]

List of Algorithms in Manopt.jl



Derivatve-Free Nelder-Mead, Particle Swarm, CMA-ES, LTMADS

Subgradient-based Subgradient Method, Convex Bundle Method, Proximal Bundle Method

Gradient-based Gradient Descent, Conjugate Gradient, Stochastic, Momentum, Nesterov, Averaged; Quasi-Newton with (L-)BFGS, DFP, Broyden, SR1,...; Levenberg-Marquard

Hessian-based Trust Regions, Adaptive Regularized Cubics (ARC)

nonsmooth Chambolle-Pock, Douglas-Rachford, Cyclic Proximal Point

constrained Augmented Lagrangian, Exact Penalty, Frank-Wolfe, Interior Point Newton

nonconvex Difference of Convex Algorithm, DCPPA

manoptjl.org/stable/solvers/