Nonsmooth Optimization on Riemannian Manifolds

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Nonsmooth Optimization on Riemannian Manifolds

We are looking for numerical algorithms to find

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} f(p)$$

where

- $\mathcal{M}$ is a Riemannian manifold
- $f\colon \mathcal{M} \to \overline{\mathbb{R}}$ is a function
- ⚠ f might be nonsmooth and/or nonconvex
- ⚠ $\mathcal{M}$ might be high-dimensional
- 💡 f has some "nice structure"

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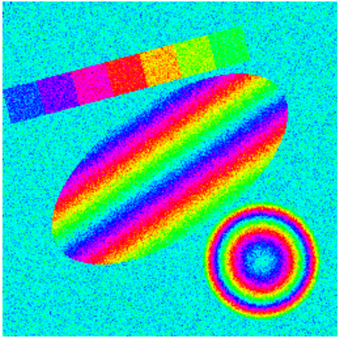
Manifold-valued signal and image processing

- variational models for
  denoising, inpainting, deconvolution, segmentation, …
- applications in medical imaging, computer vision
- nonlinear (non-Euclidean) data

Examples

- phase-valued data ($\mathbb{S}^1$)
- wind-fields, GPS ($\mathbb{S}^2$)
- DT-MRI ($\mathcal{P}(3)$)
- EBSD, (grain) orientations ($\mathrm{SO}(n)$)



Artificial noisy phase-valued data.

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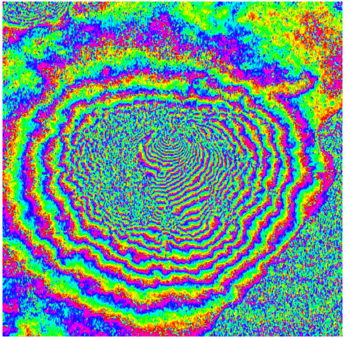

InSAR-Data of Mt. Vesuvius.
[Rocca, Prati, Guarnieri, 1997]

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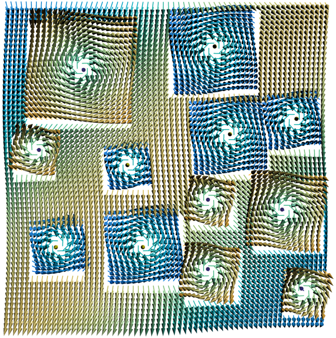

Artificial noisy data on the sphere $\mathbb{S}^2$.

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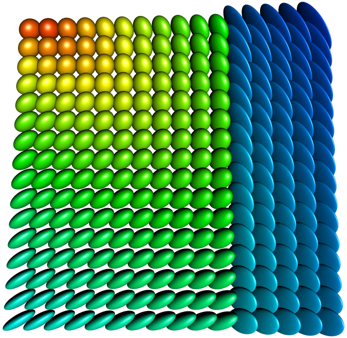

Artificial diffusion data,
each pixel is a sym. pos. def. matrix.

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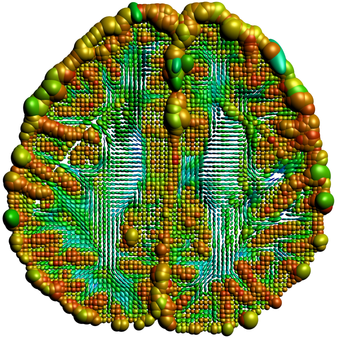

DT-MRI of the human brain.
Camino Profject: cmic.cs.ucl.ac.uk/camino

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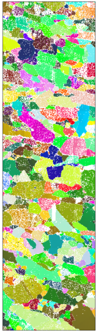


Grain orientations in EBSD data.

MTEX toolbox: mtex-toolbox.github.io

Constraints and/or geometry

constraints

- needs an embedding
- might not always yield a manifold
- 😊 slightly more flexible
- 😐 algorithms have to deal with constraints
- 😐 results might be infeasible

geometry

- might work agnostic of an embedding
- 😊 quotient manifolds
- 😊 we can use any uncinstrained algorithm...
- 😐 ...after adapting it to the manifold setting
- 😊 algorithms stay on the manifold → always feasible

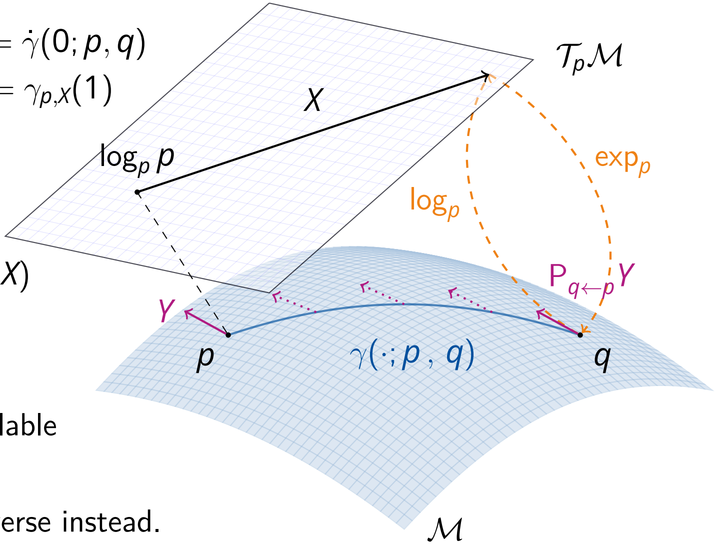
We can also consider a combination of both:
constrained optimization on manifolds.

[Liu, Boumal, 2019; RB, Herzog, 2019]

A Riemannian Manifold $\mathcal{M}$

Notation.

- Logarithmic map $\log_p q = \dot\gamma(0; p, q)$
- Exponential map $\exp_p X = \gamma_{p,X}(1)$
- Geodesic $\gamma(\cdot; p, q)$
- Tangent space $\mathcal{T}_p\mathcal{M}$
- inner product $(\cdot\,,\,\cdot)_p$
- parallel transport $\mathrm{PT}_{p\leftarrow q}(X)$
- distance function $d(p, q)$



Numerics.

$\exp_p$ and $\log_p$ maybe not available efficiently/ in closed form

$\Rightarrow$ use a retraction and its inverse instead.

(Geodesic) Convexity

[Sakai, 1996; Udriște, 1994]

A set $\mathcal{C} \subset \mathcal{M}$ is called (strongly geodesically) convex
if for all $p, q \in \mathcal{C}$ the geodesic $\gamma(\cdot; p, q)$ is unique and lies in $\mathcal{C}$.

A function $f\colon \mathcal{C} \to \overline{\mathbb{R}}$ is called (geodesically) convex
if for all $p, q \in \mathcal{C}$ the composition $f(\gamma(t; p, q)), t \in [0, 1]$, is convex.

The Riemannian Subdifferential

Let $\mathcal{C}$ be a convex set.

The subdifferential of f at $p \in \mathcal{C}$ is given by [Ferreira, Oliveira, 2002; Lee, 2003; Udriște, 1994]

$$\partial_{\mathcal{M}} f(p) := \bigl\{\xi \in \mathcal{T}_p^*\mathcal{M} \,\big|\, f(q) \geq f(p) + \langle \xi \,, \log_p q\rangle_p \ \text{ for } q \in \mathcal{C}\bigr\},$$

where

- $\mathcal{T}_p^*\mathcal{M}$ is the dual space of $\mathcal{T}_p\mathcal{M}$, also called cotangent space
- $\langle\cdot\,,\cdot\rangle_p$ denotes the duality pairing on $\mathcal{T}_p^*\mathcal{M} \times \mathcal{T}_p\mathcal{M}$
- numerically we use musical isomorphisms $X = \xi^\flat \in \mathcal{T}_p\mathcal{M}$ to obtain a subset of $\mathcal{T}_p\mathcal{M}$

The Proximal Point Algorithm

Euclidean case. For $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $\lambda > 0$, the proximal map given by

[Moreau, 1965; Rockafellar, 1970]

$$\operatorname{prox}_{\lambda f}(x) = \operatorname*{arg\,min}_{y\in\mathbb{R}^n}\Bigl\{f(y) + \frac{1}{2\lambda}\lVert y-x\rVert^2\Bigr\}.$$

Riemannian case. For $f\colon \mathcal{M} \to \overline{\mathbb{R}}$, $\lambda > 0$, the proximal map is given by

[Ferreira, Oliveira, 2002]

$$\operatorname{prox}_{\lambda f}(p) = \operatorname*{arg\,min}_{q\in\mathcal{M}}\Bigl\{f(q) + \frac{1}{2\lambda}d(p,q)^2\Bigr\}.$$

For both. A minimizer p^* of f is a fixed point for $\operatorname{prox}_{\lambda f}$.

Proximal Point Algorithm (PPA). Given $p^{(0)} \in \mathcal{M}$, $\lambda_k > 0$, iterate

$$p^{(k+1)} = \operatorname{prox}_{\lambda_k f}(p^{(k)}).$$

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The Cyclic Proximal Point Algorithm [Bertsekas, 2011; Bačák, 2014]

For a splitting $f(p) = \sum_{i=1}^{c} f_i(p)$ and some $p_0 \in \mathcal{M}$, we can use

$$p_{k+\frac{i+1}{c}} = \operatorname{prox}_{\lambda_k f_i}(p_{k+\frac{i}{c}}), \qquad i = 0, \ldots, c-1, \quad k = 0, 1, \ldots$$

On a Hadamard manifold $\mathcal{M}$: Convergence to a minimizer of f if

- all f_i proper, convex, lower semi-continuous
- $\{\lambda_k\}_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})\setminus\ell_1(\mathbb{N})$.
- also for
  - random order of the $\operatorname{prox}_{\lambda f_i}$
  - inexact evaluations of the $\operatorname{prox}_{\lambda f_i}$ [Bačák, RB, Steidl, Weinmann, 2016]

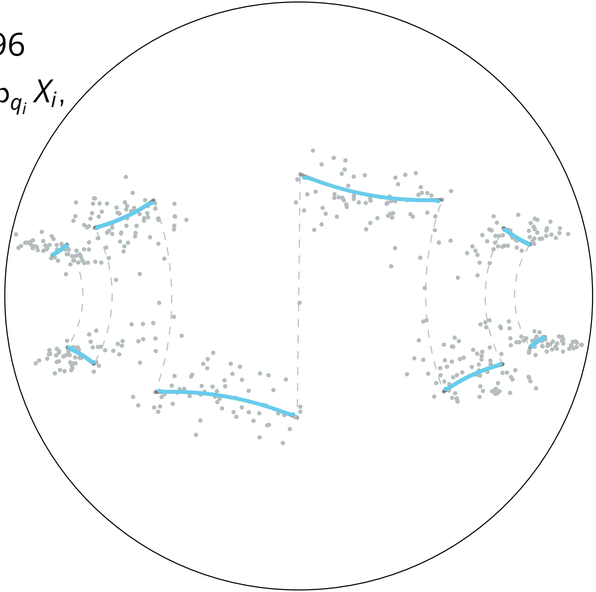
! no convergence rate

Denoising a Signal on Hyperbolic Space $\mathcal{H}^2$

- signal $q \in \mathcal{M}$, $(\mathcal{H}^2)^n$, $n = 496$
- noisy signal $\bar{q} \in \mathcal{M}$, $\bar{q}_i = \exp_{q_i} X_i$, $\sigma = 0.1$
- ROF Model:

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} \frac{1}{n} d_{\mathcal{M}}(p,q)^2 + \alpha \sum_{i=1}^{n-1} d_{\mathcal{H}^2}(p_i, p_{i+1})$$

- Setting $\alpha = 0.05$ yields reconstruction p^*.



Algorithms for Denoising a Signal

- Riemannian Convex Bundle Method (RCBM) [RB, Herzog, Jasa, 2024]
- Proximal Bundle Algorithm (PBA) [Hoseini Monjezi, Nobakhtian, Pouryayevali, 2021]
- Subgradient Method (SGM) [Ferreira, Oliveira, 1998]
- Cyclic Proximal Point Algorithm (CPPA) [Bačák, 2014]

Algorithm	Iter.	Time (sec.)	Objective	Error
RCBM	3417	51.393	1.7929×10^{-3}	3.3194×10^{-4}
PBA	15 000	102.387	1.8153×10^{-3}	4.3874×10^{-4}
SGM	15 000	99.604	1.7920×10^{-3}	3.3080×10^{-4}
CPPA	15 000	94.200	1.7928×10^{-3}	3.3230×10^{-4}

The Douglas Rachford Algorithm

For a splitting $f = g + h$, where both are possibly nonsmooth, use the reflection at the proximal map

$$R_{\lambda f}(p) = \exp_{\operatorname{prox}_{\lambda f}(p)}\bigl(-\log_{\operatorname{prox}_{\lambda f}(p)}(p)\bigr) \qquad (\text{Euclidean: } 2\operatorname{prox}_{\lambda f}(x) - x)$$

The Douglas Rachford algorithm reads for some $r^{(0)} \in \mathcal{M}$, $\eta > 0$

[RB, Persch, Steidl, 2016]

$$\begin{aligned} p^{(k)} &= R_{\eta g}(r^{(k)}) \\ q^{(k)} &= R_{\eta h}(p^{(k)}) \\ r^{(k+1)} &= \gamma(\lambda_k; r^{(k)}, q^{(k)}) \qquad (\gamma \text{ is a geodesic}) \end{aligned}$$

- converges on Hadamard manifolds if
  - g, h proper, convex, lsc.
  - $\lambda_k \in [0, 1]$ and $\sum_k \lambda_k(1 - \lambda_k) = \infty$
- ...to a fixed point of $R_{\lambda g} \circ R_{\lambda h}$ (in $r^{(k)}$)
- ...to a minimizer of f in the “shadow iterates” $\operatorname{prox}_{\eta g}(r^{(k)})$

The Fenchel Conjugate

The Fenchel conjugate of a function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is given by

$$f^*(\xi) := \sup_{x\in\mathbb{R}^n} \langle \xi, x\rangle - f(x) = \sup_{x\in\mathbb{R}^n} \begin{pmatrix} \xi \\ -1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x \\ f(x) \end{pmatrix}$$

- given $\xi \in \mathbb{R}^n$: maximize the distance between $\xi^{\mathrm{T}}\cdot$ and f
- can also be written in the epigraph

The Fenchel biconjugate reads

$$f^{**}(x) = (f^*)^*(x) = \sup_{\xi\in\mathbb{R}^n} \langle \xi , x\rangle - f^*(\xi).$$

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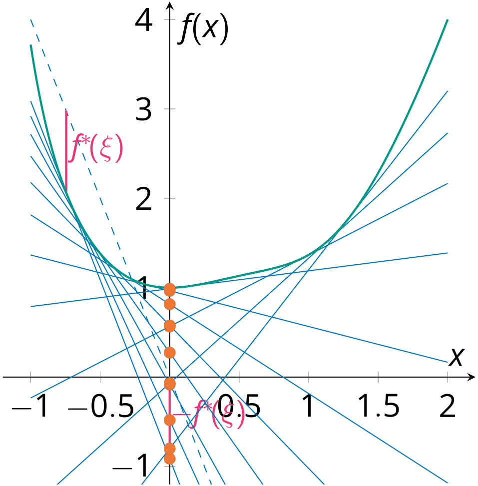
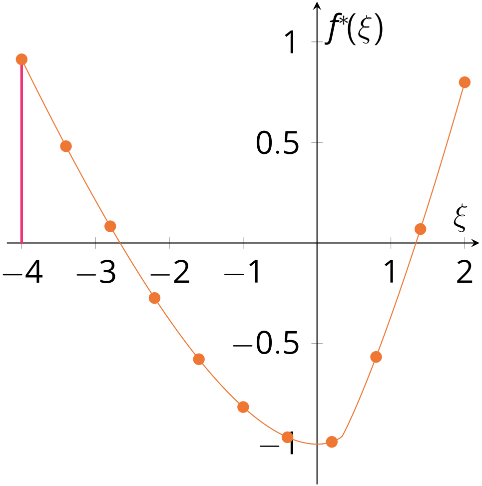
Illustration of the Fenchel Conjugate

The function f



The Fenchel conjugate f^*



The (Riemannian) m-Fenchel Conjugate

[RB, Herzog, Silva Louzeiro, Tenbrinck, Vidal-Núñez, 2021]

Idea. Localize to $\mathcal{C} \subset \mathcal{M}$ around a point m which "acts as" 0.

The m-Fenchel conjugate of a function $f\colon \mathcal{C} \to \overline{\mathbb{R}}$ is given by

$$f_m^*(\xi_m) := \sup_{X \in \mathcal{L}_{\mathcal{C},m}} \bigl\{ \langle \xi_m \,, X \rangle - f(\exp_m X) \bigr\},$$

where $\mathcal{L}_{\mathcal{C},m} := \{X \in \mathcal{T}_m\mathcal{M} \mid q = \exp_m X \in \mathcal{C} \text{ and } \|X\|_p = d(q,p)\}$.

Let $m' \in \mathcal{C}$. The mm'-Fenchel-biconjugate $F_{mm'}^{**}\colon \mathcal{C} \to \overline{\mathbb{R}}$ is given by

$$F_{mm'}^{**}(p) = \sup_{\xi_{m'} \in \mathcal{T}_{m'}^*\mathcal{M}} \bigl\{ \langle \xi_{m'} \,, \log_{m'} p \rangle - F_m^*(\mathrm{P}_{m \leftarrow m'} \xi_{m'}) \bigr\},$$

where usually we only use the case $m = m'$.

The exact Riemannian Chambolle–Pock Algorithm

[RB, Herzog, Silva Louzeiro, Tenbrinck, Vidal-Núñez, 2021; Valkonen, 2014; Chambolle, Pock, 2011]

Input: m, $p^{(0)} \in \mathcal{C} \subset \mathcal{M}$, $n = \Lambda(m)$, $\xi_n^{(0)} \in \mathcal{T}_n^*\mathcal{N}$, and $\sigma, \tau, \theta > 0$

$k \leftarrow 0$

$\bar{p}^{(0)} \leftarrow p^{(0)}$

while not converged **do**

$\qquad \xi_n^{(k+1)} \leftarrow \operatorname{prox}_{\tau g_n^*}\Bigl(\xi_n^{(k)} + \tau\bigl(\log_n \Lambda(\bar{p}^{(k)})\bigr)^\flat\Bigr)$

$\qquad p^{(k+1)} \leftarrow \operatorname{prox}_{\sigma f}\Biggl(\exp_{p^{(k)}}\Bigl(\mathrm{P}_{p^{(k)}\leftarrow m}\bigl(-\sigma D\Lambda(m)^*[\xi_n^{(k+1)}]\bigr)^\sharp\Bigr)\Biggr)$

$\qquad \bar{p}^{(k+1)} \leftarrow \exp_{p^{(k+1)}}\bigl(-\theta \log_{p^{(k+1)}} p^{(k)}\bigr)$

$\qquad k \leftarrow k + 1$

end while

Output: $p^{(k)}$

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Proximal Gradient

For a splitting $f = g + h$, where g is smooth and h is possibly nonsmooth, both are convex.
The proximal gradient method reads for given $p^{(0)} \in \mathcal{M}$, $\lambda_k \in (0, \frac{1}{L}]$ reads

[RB, Jasa, John, Pfeffer, 2025b]

$$p^{(k+1)} = \operatorname{prox}_{\lambda_k h}\bigl(\exp_{p^{(k)}}(-\lambda_k \operatorname{grad} g(p^{(k)}))\bigr).$$

- convergence rates: sublinear (convex) linear (strongly convex)
- a generalization of the prox-grad inequality
- even the nonconvex case: sublinear convergence to ε-stationary points [RB, Jasa, John, Pfeffer, 2025a]

! though here: proximal map maybe not unique minimizer

The Riemannian DC Algorithm

[RB, Ferreira, Santos, Souza, 2024]

To solve a Difference of Convex problem

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - h(p).$$

use

The Riemannian Difference of Convex Algorithm.

Input: An initial point $p^{(0)} \in \operatorname{dom}(g)$, g and $\partial_{\mathcal{M}} h$

1: Set $k = 0$.

2: **while** not converged **do**

3: Take $X^{(k)} \in \partial_{\mathcal{M}} h(p^{(k)})$

4: Compute the next iterate $p^{(k+1)}$ as

$$p^{(k+1)} \in \operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - \big(X^{(k)}, \log_{p^{(k)}} p\big)_{p^{(k)}}.$$

5: Set $k \leftarrow k + 1$

6: **end while**

Convergence of the Riemannian DCA

Let $\{p^{(k)}\}_{k\in\mathbb{N}}$ and $\{X^{(k)}\}_{k\in\mathbb{N}}$ be the iterates and subgradients of the RDCA.

Theorem. [RB, Ferreira, Santos, Souza, 2024]

If $\bar{p}$ is a cluster point of $\{p^{(k)}\}_{k\in\mathbb{N}}$, then $\bar{p} \in \operatorname{dom}(g)$ and there exists a cluster point $\bar{X}$ of $\{X^{(k)}\}_{k\in\mathbb{N}}$ s. t. $\bar{X} \in \partial g(\bar{p}) \cap \partial h(\bar{p})$.

$\Rightarrow$ Every cluster point of $\{p^{(k)}\}_{k\in\mathbb{N}}$, if any, is a critical point of f.

Proposition. [RB, Ferreira, Santos, Souza, 2024]

Let g be σ-strongly (geodesically) convex. Then

$$f(p^{(k+1)}) \leq f(p^{(k)}) - \frac{\sigma}{2} d^2(p^{(k)}, p^{(k+1)})$$

and $\displaystyle\sum_{k=0}^{\infty} d^2(p^{(k)}, p^{(k+1)}) < \infty$, so in particular $\displaystyle\lim_{k\to\infty} d(p^{(k)}, p^{(k+1)}) = 0$.

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Software

Goals of the Software – Why Julia?

Goals.

- abstract definition of manifolds and optimization thereon

⇒ implement abstract solvers on a generic manifold

- well-documented and well-tested
- fast.

⇒ "Run your favourite solver on your favourite manifold".

Why Julia? julialang.org

- high-level language, properly typed
- multiple dispatch, e. g. `*(::AbstractMatrix, ::AbstractMatrix)`
- just-in-time compilation, solves two-language problem
  ⇒ "nice to write" and as fast as C/C++
- I like the community

ManifoldsBase.jl – Motivation

Goal. Provide a generic interface to manifolds for

- defining own (new) manifolds
- implementing generic algorithms on an arbitrary manifold $\mathcal{M}$

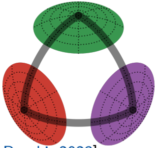
A Manifold. a Riemannian manifold is a subtype of `AbstractManifold{`$\mathbb{F}$`}`

- $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$: field the manifold is build on
- stores all "general" information, (mainly) the manifold dimension
- example (form `Manifolds.jl`): `M = Sphere(2)`

Points and Tangent vectors.

- by default not typed, often `<:AbstractArray`
- we provide `<:AbstractManifoldPoint` and `<:TVector` for more advanced ones

Manifolds.jl



[Axen, Baran, RB, Rzecki, 2023]

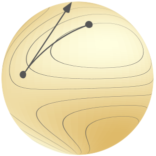
Goal. Provide a library of Riemannian manifolds, that is efficiently implemented and well-documented

Meta. generic implementations for $\mathcal{M}^{n \times m}$, $\mathcal{M}_1 \times \mathcal{M}_2$, vector- and tangent-bundles, esp. $T_p\mathcal{M}$, or Lie groups

Library. Implemented functions for

- Circle, Sphere, Torus, Hyperbolic, Projective Spaces, Hamiltonian
- (generalized, symplectic) Stiefel, Rotations
- (generalized, symplectic) Grassmann, fixed rank matrices
- Symmetric Positive Definite matrices, with fixed determinant
- (several) Multinomial, (skew-)symmetric, and symplectic matrices
- Tucker & Oblique manifold, Kendall's Shape space
- probability simplex, orthogonal and unitary matrices, …

Manopt.jl



Goal. Provide optimization algorithms on Riemannian manifolds.

Features. Given a `Problem p` and a `SolverState s`, implement `initialize_solver!(p, s)` and `step_solver!(p, s, i)`
⇒ an algorithm in the `Manopt.jl` interface

Highlevel interfaces like `gradient_descent(M, f, grad_f)` on any manifold `M` from `Manifolds.jl`.

All provide debug output, recording, cache & counting capabilities, as well as a library of step sizes and stopping criteria.

Manopt family.

manoptjl.org
[RB, 2022]

manopt.org
[Boumal, Mishra, Absil, Sepulchre, 2014]

pymanopt.org
[Townsend, Koep, Weichwald, 2016]

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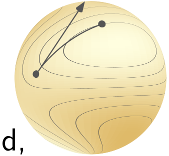
List of Algorithms in Manopt.jl



Derivatve-Free Nelder-Mead, Particle Swarm, CMA-ES, MADS

Subgradient-based Subgradient Method, Convex Bundle Method, Proximal Bundle Method

Gradient-based Gradient Descent, Conjugate Gradient, Stochastic, Momentum, Nesterov, Averaged; Quasi-Newton with (L-)BFGS, DFP, Broyden, SR1,...; Levenberg-Marquard

Hessian-based Trust Regions, Adaptive Regularized Cubics (ARC)

splitting Chambolle-Pock, Douglas-Rachford, Cyclic Proximal Point, Proximal Gradient

constrained Augmented Lagrangian, Exact Penalty, Frank-Wolfe, Projected Gradient, Interior Point Newton

nonconvex Difference of Convex Algorithm, DCPPA

manoptjl.org/stable/solvers/

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A Numerical Example

The Difference of Convex Algorithm in Manopt.jl

The algorithm is implemented and released in Julia using `Manopt.jl`[1].
It can be used with any manifold from `Manifolds.jl`

A solver call looks like

```
q = difference_of_convex_algorithm(M, f, g, ∂h, p0)
```

where one has to implement `f(M, p)`, `g(M, p)`, and `∂h(M, p)`.

- a sub problem is generated if keyword `grad_g=` is set
- an efficient version of its cost and gradient is provided
- you can specify the sub-solver using `sub_state=`
  to also set up the specific parameters of your favourite algorithm

[1] see https://manoptjl.org/stable/solvers/difference_of_convex/

Rosenbrock and First Order Methods

Problem. We consider the classical Rosenbrock example[2]

$$\operatorname*{arg\,min}_{x\in\mathbb{R}^2} a\bigl(x_1^2 - x_2\bigr)^2 + \bigl(x_1 - b\bigr)^2,$$

where $a, b > 0$, usually $b = 1$ and $a \gg b$, here: $a = 2 \cdot 10^5$.

Known Minimizer $x^* = \begin{pmatrix} b \\ b^2 \end{pmatrix}$ with cost $f(x^*) = 0$.

Goal. Compare first-order methods, e. g. using the (Euclidean) gradient

$$\nabla f(x) = \begin{pmatrix} 4a(x_1^2 - x_2) \\ -2a(x_1^2 - x_2) \end{pmatrix} + \begin{pmatrix} 2(x_1 - b) \\ 0 \end{pmatrix}$$

[2] available online in ManoptExamples.jl

A "Rosenbrock-Metric" on $\mathbb{R}^2$

In our Riemannian framework, we can introduce a new metric on $\mathbb{R}^2$ as

$$G_p := \begin{pmatrix} 1+4p_1^2 & -2p_1 \\ -2p_1 & 1 \end{pmatrix}, \text{ with inverse } G_p^{-1} = \begin{pmatrix} 1 & 2p_1 \\ 2p_1 & 1+4p_1^2 \end{pmatrix}.$$

We obtain $(X,Y)_p = X^{\mathrm{T}} G_p Y$

The exponential and logarithmic map are given as

$$\exp_p(X) = \begin{pmatrix} p_1 + X_1 \\ p_2 + X_2 + X_1^2 \end{pmatrix}, \qquad \log_p(q) = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 - (q_1 - p_1)^2 \end{pmatrix}.$$

`Manifolds.jl`:
Implement these functions on `MetricManifold(ℝ^2, RosenbrockMetric())`.

The Riemannian Gradient w.r.t. the new Metric

Let $f\colon \mathcal{M} \to \mathbb{R}$. Given the Euclidean gradient $\nabla f(p)$, its Riemannian gradient $\operatorname{grad} f\colon \mathcal{M} \to T\mathcal{M}$ is given by

$$\operatorname{grad} f(p) = G_p^{-1} \nabla f(p).$$

While we could implement this denoting $\nabla f(p) = \bigl(f_1'(p) \quad f_2'(p)\bigr)^{\mathrm{T}}$ using

$$\Bigl\langle \operatorname{grad} f(q), \log_q p \Bigr\rangle_q = (p_1 - q_1) f_1'(q) + (p_2 - q_2 - (p_1 - q_1)^2) f_2'(q),$$

but it is automatically done in `Manopt.jl`.

The Experiment Setup

Algorithms. We now compare

1. The Euclidean gradient descent algorithm on $\mathbb{R}^2$,
2. The Riemannian gradient descent algorithm on $\mathcal{M}$,
3. The Difference of Convex Algorithm on $\mathbb{R}^2$,
4. The Difference of Convex Algorithm on $\mathcal{M}$.

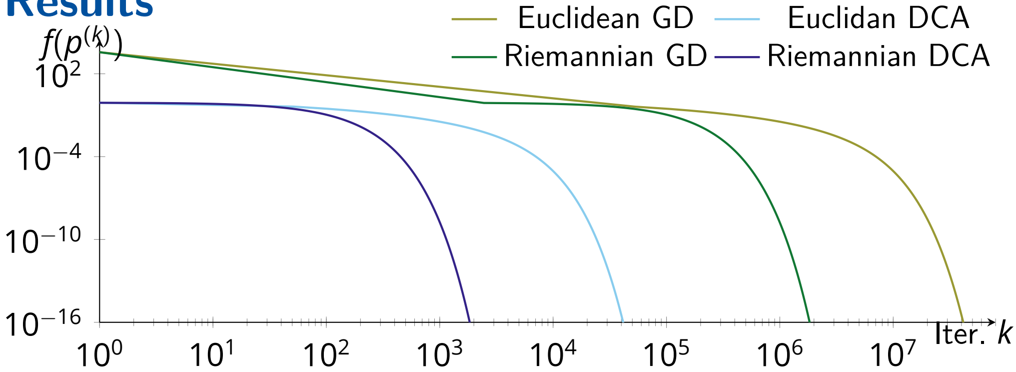
For DCA third we split f into $f(x) = g(x) - h(x)$ with

$$g(x) = a\left(x_1^2 - x_2\right)^2 + 2\left(x_1 - b\right)^2 \quad \text{and} \quad h(x) = \left(x_1 - b\right)^2.$$

Initial point. $p_0 = \frac{1}{10}\begin{pmatrix}1\\2\end{pmatrix}$ with cost $f(p_0) \approx 7220.81$.

Stopping Criterion.
$d_{\mathcal{M}}(p^{(k)}, p^{(k-1)}) < 10^{-16}$ or $\|\operatorname{grad} f(p^{(k)})\|_p < 10^{-16}$.

Results



Algorithm	**Runtime** (sec.)	**# Iterations**
Euclidean GD	305.567	53 073 227
Euclidean DCA	58.268	50 588
Riemannian GD	18.894	2 454 017
Riemannian DCA	7.704	2 459

Summary

Nonsmooth optimization on manifolds appears in several applications.

- many algorithms can be generalized
- many properties carry over, like convergence results
- Fenchel duality can be generalized [Schiela, Herzog, RB, 2024]
- `Manifolds.jl` & `Manopt.jl` [RB, 2022; Axen, Baran, RB, Rzecki, 2023]
- numerical examples available in `ManoptExamples.jl`

- **Next.** `LieGroups.jl`

Selected References

Axen, S. D.; M. Baran; RB; K. Rzecki (2023). "Manifolds.jl: An Extensible Julia Framework for Data Analysis on Manifolds". *ACM Transactions on Mathematical Software* 49.4. DOI: 10.1145/3618296.

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RB; J. Persch; G. Steidl (2016). "A parallel Douglas Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifolds". *SIAM Journal on Imaging Sciences* 9.4, pp. 901–937. DOI: 10.1137/15M1052858.