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Nonsmooth Optimization on Riemannian manifolds

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joint work with

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Nonsmooth Optimization on Riemannian Manifolds

We are looking for numerical algorithms to find

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} f(p)$$

where

- $\mathcal{M}$ is a Riemannian manifold
- $f\colon \mathcal{M} \to \overline{\mathbb{R}}$ is a function

⚠ f might be nonsmooth and/or nonconvex

⚠ $\mathcal{M}$ might be high-dimensional

A Riemannian Manifold $\mathcal{M}$

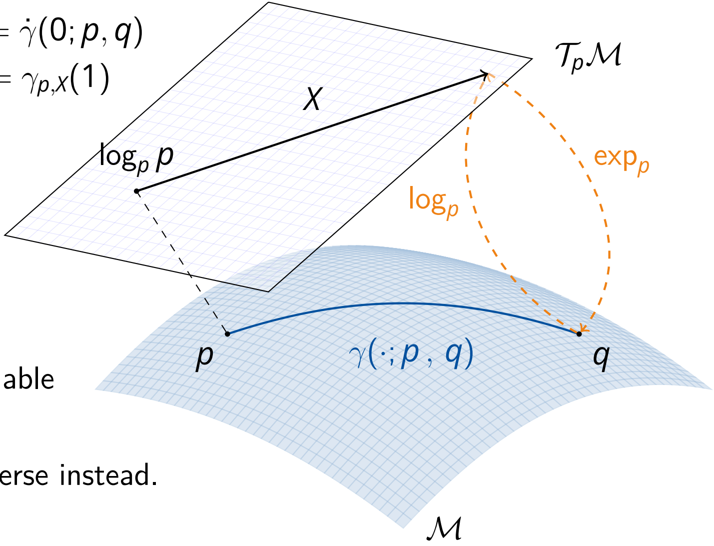
A d-dimensional Riemannian manifold can be informally defined as a set $\mathcal{M}$ covered with a "suitable" collection of charts, that identify subsets of $\mathcal{M}$ with open subsets of $\mathbb{R}^d$ and a continuously varying inner product on the tangent spaces.

[Absil, Mahony, and Sepulchre 2008]

A Riemannian Manifold $\mathcal{M}$

Notation.

- Logarithmic map $\log_p q = \dot\gamma(0; p, q)$
- Exponential map $\exp_p X = \gamma_{p,X}(1)$
- Geodesic $\gamma(\cdot; p\,,\,q)$
- Tangent space $\mathcal{T}_p\mathcal{M}$
- inner product $(\cdot\,,\,\cdot)_p$



Numerics.

$\exp_p$ and $\log_p$ maybe not available efficiently/ in closed form

$\Rightarrow$ use a retraction and its inverse instead.

(Geodesic) Convexity

[Sakai 1996; Udriște 1994]

A set $\mathcal{C} \subset \mathcal{M}$ is called (strongly geodesically) convex
if for all $p, q \in \mathcal{C}$ the geodesic $\gamma(\cdot; p\,, q)$ is unique and lies in $\mathcal{C}$.

A function $f\colon \mathcal{C} \to \overline{\mathbb{R}}$ is called (geodesically) convex
if for all $p, q \in \mathcal{C}$ the composition $f(\gamma(t; p\,, q)), t \in [0, 1]$, is convex.

The Riemannian Subdifferential

Let $\mathcal{C}$ be a convex set.

The subdifferential of f at $p \in \mathcal{C}$ is given by [O. Ferreira and Oliveira 2002; Lee 2003; Udriște 1994]

$$\partial_{\mathcal{M}} f(p) := \Bigl\{ \xi \in \mathcal{T}_p^* \mathcal{M} \,\Big|\, f(q) \geq f(p) + \langle \xi \,, \log_p q \rangle_p \ \text{ for } q \in \mathcal{C} \Bigr\},$$

where

- $\mathcal{T}_p^*\mathcal{M}$ is the dual space of $\mathcal{T}_p\mathcal{M}$, also called cotangent space
- $\langle \cdot \,, \cdot \rangle_p$ denotes the duality pairing on $\mathcal{T}_p^*\mathcal{M} \times \mathcal{T}_p\mathcal{M}$

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The Riemannian Convex Bundle Method

The ε-Subdifferential

Let $\varepsilon > 0$.
The $\varepsilon-$subdifferential of a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$ reads

$$\partial_\varepsilon f(x) = \left\{ s \in \mathbb{R}^n \,\middle|\, f(y) \geq f(x) + s^\mathrm{T}(y - x) - \varepsilon \text{ for all } y \in \mathbb{R}^n \right\}$$

Let $\varepsilon > 0$ and $\mathcal{C} \subset \mathcal{M}$ be a convex set.
The $\varepsilon-$subdifferential of a convex function $f\colon \mathcal{C} \to \mathbb{R}$ reads

$$\partial_\varepsilon f(x) = \left\{ X \in \mathcal{T}_p\mathcal{M} \,\middle|\, f(q) \geq f(p) + (X, \log_p q) - \varepsilon \text{ for all } q \in \mathcal{C} \right\}$$

Clearly in both cases $\partial f(x) = \partial_0 f(x) \subset \partial_\varepsilon f(x)$

The Riemannian Convex Bundle Method

[RB, Herzog, and Jasa 2024]

- Given $f\colon \mathcal{C} \to \mathbb{R}$ on a (geodesically) convex set $\mathcal{C} \subset \mathcal{M}$
- collect
  - subgradients $X_{q^{(k)}} \in \partial f(q^{(k)})$
  - stabilisation centers $p^{(k)}$ ("best" iterates)
- use this information to
  - determine the next descent direction $d^{(k)} \in \mathcal{T}_{p^{(k)}}\mathcal{M}$ by solving a QP in $\mathcal{T}_{p^{(k)}}\mathcal{M}$
  - where $d^{(k)} \in \partial_{c^{(k)}} f(p^{(k)})$
- we stop when both
  - the approximation $\partial_{c^{(k)}} f(p^{(k)})$ of $\partial f(p^{(k)})$ is "good enough"
  - $\|d^{(k)}\|$ is "small enough"

Aproximating the ε-Subdifferential

For $f\colon \mathbb{R}^n \to \mathbb{R}$, given $x^{(0)}, \ldots, x^{(k)} \in \mathbb{R}^n$, and $s^{(j)} \in \partial f(x^{(j)})$,
define the linearization errors

$$e_j^{(k)} := f(x^{(k)}) - f(x^{(j)}) - (s^{(j)})^T(x^{(k)} - x^{(j)}), \qquad j = 0, \ldots, k.$$

Then (Geiger and Kanzow 2002, Theorem 6.68)

$$s^{(j)} \in \partial_{e_j^{(k)}} f(x^{(k)})$$

and we can characterize an inner approximation $G_\varepsilon^{(k)} \subseteq \partial_\varepsilon f(x^{(k)})$ as

$$G_\varepsilon^{(k)} := \left\{ \sum_{j=0}^{k} \lambda_j s^{(j)} \;\middle|\; \sum_{j=0}^{k} \lambda_j\, e_j^{(k)} \leq \varepsilon,\ \sum_{j=0}^{k} \lambda_j = 1,\ \lambda_j \geq 0 \text{ for all } j = 0, \ldots, k \right\}$$

Challenge on manifolds.
How can we take into account curvature in the error terms?

Curvature Correction

Let $\Omega \in \mathbb{R}$ be an upper bound on the curvature. Define

[RB, Herzog, and Jasa 2024]

$$c_j^{(k)} := f(p^{(k)}) - f(p^{(j)}) - \left(X_{p^{(j)}} \,,\, \log_{p^{(j)}} p^{(k)} \right) \quad \text{if } \Omega \leq 0,$$
$$c_j^{(k)} := f(p^{(k)}) - f(p^{(j)}) + \|X_{p^{(j)}}\| \|\log_{p^{(j)}} p^{(k)}\| \quad \text{if } \Omega > 0.$$

Then we get

$$G_\varepsilon^{(k)} := \left\{ \sum_{j=0}^{k} \lambda_j \mathrm{P}_{p^{(k)} \leftarrow p^{(j)}} X_{p^{(j)}} \,\middle|\, \sum_{j=0}^{k} \lambda_j \, e_j^{(k)} \leq \varepsilon, \ \sum_{j=0}^{k} \lambda_j = 1, \ \lambda_j \geq 0, j = 0, \ldots, k \right\}$$

with $G_\varepsilon^{(k)} \subseteq \partial_\varepsilon f(p^{(k)})$, and $\mathrm{P}_{p^{(k)} \leftarrow p^{(j)}} X_{p^{(j)}} \in \partial_{c_j^{(k)}} f(p^{(k)})$.

The Riemannian Subproblem

Let $k \in \mathbb{N}$ and $j \in \{0, \dots, k\} = J^{(k)}$ and $X_{p^{(j)}} \in \partial f(p^{(j)})$, $p^{(j)} \in \mathbb{R}^n$

For a coefficients $\lambda_j \geq 0$ with $\sum_j \lambda_j = 1$, we have

$$\sum_{j \in J^{(k)}} \lambda_j \mathrm{P}_{p^{(k)} \leftarrow q^{(j)}} X_{q^{(j)}} \in \partial_\varepsilon f(p^{(k)}) \qquad \text{if and only if} \qquad \sum_{j \in J^{(k)}} \lambda_j c_j^{(k)} \leq \varepsilon$$

Solving the constrained quadratic problem

$$\begin{aligned} \operatorname*{arg\,min}_{\lambda \in \mathbb{R}^{|J^{(k)}|}} \quad & \frac{1}{2} \Bigl\lVert \sum_{j \in J^{(k)}} \lambda_j \mathrm{P}_{p^{(k)} \leftarrow q^{(j)}} X_{q^{(j)}} \Bigr\rVert^2 + \sum_{j \in J^{(k)}} \lambda_j e_j^{(k)} c_j^{(k)} \\ \text{s. t.} \quad & \sum_{j \in J^{(k)}} \lambda_j = 1, \quad \lambda_j \geq 0 \quad \text{for all } j \in J^{(k)} \end{aligned}$$

yields the new search direction

$$d^{(k)} := - \sum_{j \in J^{(k)}} \lambda_j \mathrm{P}_{p^{(k)} \leftarrow q^{(j)}} X_{q^{(j)}}.$$

The Riemannian Convex Bundle Method

Input: $p^{(0)} = q^{(0)} \in \mathcal{C}$, $g^{(0)} = X_{p_0} \in \partial f(p^{(0)})$, $m \in (0,1)$,
$\varepsilon^{(0)} = e^{(0)} c^{(0)} = 0$, $J^{(0)} = \{0\}$, and $k = 0$.

1: **while** not converged **do**
2: Set $k = k + 1$
3: Compute a solution $\lambda^{(k)} \in \mathbb{R}^{|J^{(k)}|}$ of the subproblem.
4: Set
$$g^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} \mathrm{P}_{p^{(k)} \leftarrow q^{(j)}} X_{q^{(j)}}, \qquad \varepsilon^{(k)} := \sum_{j \in J^{(k)}} \lambda_j^{(k)} e_j^{(k)} c_j^{(k)},$$
$$d^{(k)} := -g^{(k)}, \qquad \xi^{(k)} := -\|g^{(k)}\|^2 - \varepsilon^{(k)},$$
5: Set $q^{(k+1)} = \exp_{p^{(k)}} d^{(k)}$ [1] and take $X_{q^{(k+1)}} \in \partial f(q^{(k+1)})$,
6: If $f(q^{(k+1)}) \leq f(p^{(k)}) + m\xi^{(k)}$ set $p^{(k+1)} = q^{(k+1)}$ else $p^{(k+1)} = p^{(k)}$
7: Update $J^{(k+1)} = \{j \in J^{(k)} \mid \lambda_j^{(k)} > 0\} \cup \{k+1\}$, and $c_j^{(k+1)}$
8: **end while**

Output: $p^{(k_*)}$ from the final $k_* \in \mathbb{N}$.

[1] Perform a backtracking if $q^{(k+1)} \notin \operatorname{int}(\operatorname{dom} f)$ or equal to $p^{(k)}$

Convergence

Theorem (Geiger and Kanzow 2002, Theorem 6.80)
Let the solution set $S = \{x^* \in \mathbb{R}^n \,|\, f(x^*) = \inf_{x\in\mathbb{R}^n} f(x)\}$ of the minimization problem be nonempty. Then every sequence $\{x^{(k)}\}$ generated by the bundle method algorithm converges to a minimizer of f.

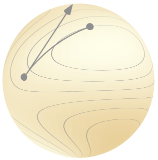
On Hadamard manifolds ($\Omega \leq 0$) we have the analogous, if

[RB, Herzog, and Jasa 2024]

1. the backtracking step size $t^{(k)} > m$ for all $k \geq k_*$, if a finite number of serious steps k_* occur
2. no accumulation point of $p^{(k)}$ is allowed to lie on the boundary of $\mathcal{C}$

Numerical Examples

Manopt.jl



Goal. Provide optimization algorithms on Riemannian manifolds.

Features. Given a `Problem p` and a `SolverState s`,
implement `initialize_solver!(p, s)` and `step_solver!(p, s, i)`
$\Rightarrow$ an algorithm in the `Manopt.jl` interface

Highlevel interfaces like `gradient_descent(M, f, grad_f)`
on any manifold `M` from `Manifolds.jl`.

All provide debug output, recording, cache & counting capabilities,
as well as a library of step sizes and stopping criteria.

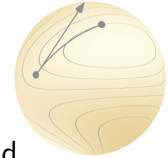
Manopt family.

manoptjl.org [RB 2022]

manopt.org [Boumal, Mishra, Absil, and Sepulchre 2014]

pymanopt.org [Townsend, Koep, and Weichwald 2016]

List of Algorithms in Manopt.jl



Derivatve Free Nelder-Mead, Particle Swarm, CMA-ES

Subgradient-based Subgradient Method, Convex Bundle Method, Proximal Bundle Method

Gradient-based Gradient Descent, Conjugate Gradient, Stochastic, Momentum, Nesterov, Averaged, ...
Quasi-Newton with (L-)BFGS, DFP, Broyden, SR1,...
Levenberg-Marquard

Hessian-based Trust Regions, Adaptive Regularized Cubics (ARC)

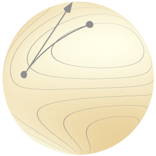
nonsmooth Chambolle-Pock, Douglas-Rachford, Cyclic Proximal Point

constrained Augmented Lagrangian, Exact Penalty, Frank-Wolfe

nonconvex Difference of Convex Algorithm, DCPPA

manoptjl.org/stable/solvers/

The Convex Bundle Method in Manopt.jl



In `Manopt.jl` a solver call looks like[2]

```
p = convex_bundle_method(M, f, ∂f, p0;
        diameter = δ, k_max = Ω, m = 10⁻³, kwargs...
    )
```

where

- `M` is a Riemannian manifold
- `f` is the objective function
- `∂f` is a subgradient of the objective function
- `p0` is an initial point on the manifold

The default stopping criterion for the algorithm is set to

$$-\xi^{(k)} \leq 10^{-8}.$$

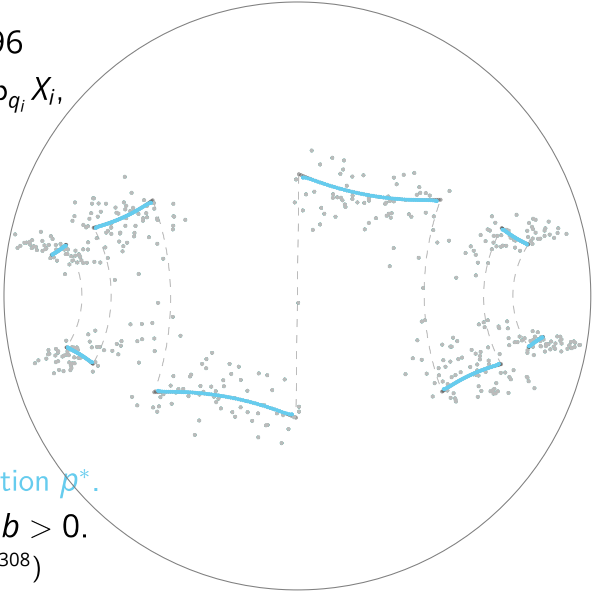
[2] full documentation: manoptjl.org/stable/solvers/convex_bundle_method/

Denoising a Signal on Hyperbolic Space $\mathcal{H}^2$

- signal $q \in \mathcal{M}$, $(\mathcal{H}^2)^n$, $n = 496$
- noisy signal $\bar{q} \in \mathcal{M}$, $\bar{q}_i = \exp_{q_i} X_i$, $\sigma = 0.1$
- ROF Model:

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} \frac{1}{n} \mathrm{d}_{\mathcal{M}}(p,q)^2 + \alpha \sum_{i=1}^{n-1} \mathrm{d}_{\mathcal{H}^2}(p_i, p_{i+1})$$

- Setting $\alpha = 0.05$ yields reconstruction p^*.
- in RCBM: set $\operatorname{diam}(\operatorname{dom} f) = b > 0$.
  (in practice: $b =$`floatmax()`$\approx 10^{308}$)



Algorithms for Denoising a Signal

- Riemannian Convex Bundle Method (RCBM) [RB, Herzog, and Jasa 2024]
- Proximal Bundle Algorithm (PBA) [Hoseini Monjezi, Nobakhtian, and Pouryayevali 2021]
- Subgradient Method (SGM) [O. Ferreira and Oliveira 1998]
- Cyclic Proximal Point Algorithm (CPPA) [Bačák 2014]

Algorithm	Iter.	Time (sec.)	Objective	Error
RCBM	3417	51.393	1.7929×10^{-3}	3.3194×10^{-4}
PBA	15 000	102.387	1.8153×10^{-3}	4.3874×10^{-4}
SGM	15 000	99.604	1.7920×10^{-3}	3.3080×10^{-4}
CPPA	15 000	94.200	1.7928×10^{-3}	3.3230×10^{-4}

The Riemanniann Median on $\mathcal{S}^d$

- Consider the $d-$dimensional sphere $\mathcal{M} = \mathcal{S}^d$
- $\bar{p}$ north pole
- $B_r(p)$ (geodesic) ball around p with radius r.
- $n = 1000$ Gaussian random data points $q^{(1)}, \ldots, q^{(n)} \in B_{\frac{\pi}{8}}(\bar{p})$
- Riemannian median on $B_{\frac{\pi}{8}}(\bar{p})$:

$$f(p) = \begin{cases} \frac{1}{n}\sum_{j=1}^{n} d_{\mathcal{M}}(p, q^{(j)}) & \text{if } p \in B_{\frac{\pi}{8}}(\bar{p}), \\ +\infty & \text{otherwise.} \end{cases}$$

Solve

$$p^* := \operatorname*{arg\,min}_{p \in \mathcal{S}^d} f(p)$$

for different manifold-dimensions d.

Algorithms for the Riemanniann Median on $\mathcal{S}^d$

	RCBM			PBA		
Dimension	Iter.	Time (sec.)	Objective	Iter.	Time (sec.)	Objective
2	19	6.50×10^{-3}	0.192 89	20	5.30×10^{-3}	0.192 89
4	28	1.01×10^{-2}	0.198 81	23	5.99×10^{-3}	0.198 81
32	58	2.29×10^{-2}	0.195 76	28	1.13×10^{-2}	0.195 76
1024	48	3.91×10^{-1}	0.197 75	40	3.31×10^{-1}	0.197 75
32 768	43	7.54	0.192 90	21	4.16	0.192 90

	SGM		
Dimension	Iter.	Time (sec.)	Objective
2	5000	1.14	0.192 89
4	3270	8.09×10^{-1}	0.198 81
32	5000	2.18	0.195 76
1024	122	9.75×10^{-1}	0.197 75
32 768	172	5.25×10^{1}	0.192 90

The Riemannian Difference of Convex Algorithm

Difference of Convex

We aim to solve

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} f(p)$$

where

- $\mathcal{M}$ is a Riemannian manifold
- $f\colon \mathcal{M} \to \mathbb{R}$ is a difference of convex function, i. e. of the form

$$f(p) = g(p) - h(p)$$

- $g, h\colon \mathcal{M} \to \overline{\mathbb{R}}$ are convex, lower semicontinuous, and proper

The Euclidean DCA

Idea 1. At x_k, approximate $h(x)$ by its affine minorization

$$h_k(x) \coloneqq h(x^{(k)}) + \langle x - x^{(k)}, y^{(k)} \rangle \text{ for some } y^{(k)} \in \partial h(x^k)$$

$$\Rightarrow \text{ iteratively minimize } g(x) - h_k(x) = g(x) - h(x^{(k)}) - \langle x - x^{(k)}, y^{(k)} \rangle$$

Idea 2. Using duality theory finding a new $y^{(k)} \in \partial h(x^{(k)})$ is equivalent to

$$y^{(k)} \in \operatorname*{arg\,min}_{y \in \mathbb{R}^n} \Bigl\{ h^*(y) - g^*(y^{(k-1)}) - \langle y - y^{(k-1)}, x^{(k)} \rangle \Bigr\}$$

Idea 3. Reformulate 2 using a proximal map $\Rightarrow$ DCPPA
on manifolds this was done in [Almeida, Neto, Oliveira, and Souza 2020; Souza and Oliveira 2015]

In the Euclidean case, all three models are equivalent.

A Fenchel Duality on a Hadamard Manifold

Let

- $T\mathcal{M} = \dot{\bigcup}_p T_p\mathcal{M}$ denote the tangent bundle
- analogously $T^*\mathcal{M}$ denotes the cotangent bundle
- $\mathcal{M}$ be a Hadamard manifold (non-positive sectional curvature).

Definition [Silva Louzeiro, RB, and Herzog 2022]

Let $f\colon \mathcal{M} \to \overline{\mathbb{R}}$.

The Fenchel conjugate of f is the function $f^*\colon \mathcal{T}^*\mathcal{M} \to \overline{\mathbb{R}}$ defined by

$$f^*(p,\xi) := \sup_{q\in\mathcal{M}} \Bigl\{ \langle \xi, \log_p q \rangle - f(q) \Bigr\}, \qquad (p,\xi) \in \mathcal{T}^*\mathcal{M}.$$

The Dual Difference of Convex Problem

Given the Difference of Convex problem

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - h(p)$$

and the Fenchel duals g^* and h^*,
we can state the dual difference of convex problem as

[RB, O. P. Ferreira, Santos, and Souza 2024]

$$\operatorname*{arg\,min}_{(p,\xi)\in T^*\mathcal{M}} \; h^*(p,\xi) - g^*(p,\xi).$$

On $\mathcal{M} = \mathbb{R}^n$ this indeed simplifies to the classical dual problem.

Theorem. [RB, O. P. Ferreira, Santos, and Souza 2024]

$$\inf_{(q,X)\in\mathcal{T}^*\mathcal{M}} \Bigl\{h^*(q,X) - g^*(q,X)\Bigr\} = \inf_{p\in\mathcal{M}} \{g(p) - h(p)\}\,.$$

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The Dual Difference of Convex Problem

The primal and dual Difference of Convex problem

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - h(p) \qquad \text{and} \qquad \operatorname*{arg\,min}_{(p,\xi)\in T^*\mathcal{M}} h^*(p,\xi) - g^*(p,\xi)$$

are equivalent in the following sense.

Theorem. [RB, O. P. Ferreira, Santos, and Souza 2024]

If p^* is a solution of the primal problem, then $(p^*, \xi^*) \in T^*\mathcal{M}$ is a solution for the dual problem for all $\xi^* \in \partial_{\mathcal{M}} h(p^*) \cap \partial_{\mathcal{M}} g(p^*)$.

If $(p^*, \xi^*) \in T^*\mathcal{M}$ is a solution of the dual problem for some $\xi^* \in \partial_{\mathcal{M}} h(p^*) \cap \partial_{\mathcal{M}} g(p^*)$, then p^* is a solution of the primal problem.

Derivation of the Riemannian DCA

We consider the first order Taylor approximation of h at some point $p^{(k)}$:
With $\xi \in \partial h(p^{(k)})$ we set

$$h_k(p) := h(p^{(k)}) + \langle \xi \,, \log_{p^{(k)}} p \rangle_{p^{(k)}}$$

Using musical isomorphisms we identify $X = \xi^\sharp \in T_p\mathcal{M}$,
where we call X a subgradient. Locally h_k minorizes h, i. e.

$$h_k(q) \leq h(q) \quad \text{locally around} \quad p^{(k)}$$

$\Rightarrow$ Use $-h_k(p)$ as upper bound for $-h(p)$ in $f = g - h$.

Note. On $\mathbb{R}^n$ the function h_k is linear.
On a manifold h_k is nonlinear and not even necessarily convex, even on a Hadamard manifold.

The Riemannian DC Algorithm

[RB, O. P. Ferreira, Santos, and Souza 2024]

Input: An initial point $p^{(0)} \in \operatorname{dom}(g)$, g and $\partial_{\mathcal{M}} h$

Set $k = 0$.
while not converged **do**
    Take $X^{(k)} \in \partial_{\mathcal{M}} h(p^{(k)})$
    Compute the next iterate $p^{(k+1)}$ as

$$p^{(k+1)} \in \operatorname*{arg\,min}_{p \in \mathcal{M}} g(p) - \bigl(X^{(k)}\,,\, \log_{p^{(k)}} p\bigr)_{p^{(k)}}. \qquad (*)$$

    Set $k \leftarrow k + 1$
end while

Note. In general the subproblem $(*)$ can not be solved in closed form. But an approximate solution yields a good candidate.

For example: Given g, $p^{(k)}$, and $X^{(k)}$ and $\operatorname{grad} g \Rightarrow$ Gradient descent.

Convergence of the Riemannian DCA

Let $\{p^{(k)}\}_{k\in\mathbb{N}}$ and $\{X^{(k)}\}_{k\in\mathbb{N}}$ be the iterates and subgradients of the RDCA.

Theorem. [RB, O. P. Ferreira, Santos, and Souza 2024]

If $\bar{p}$ is a cluster point of $\{p^{(k)}\}_{k\in\mathbb{N}}$, then $\bar{p} \in \operatorname{dom}(g)$ and there exists a cluster point $\bar{X}$ of $\{X^{(k)}\}_{k\in\mathbb{N}}$ s. t. $\bar{X} \in \partial g(\bar{p}) \cap \partial h(\bar{p})$.

$\Rightarrow$ Every cluster point of $\{p^{(k)}\}_{k\in\mathbb{N}}$, if any, is a critical point of f.

Proposition. [RB, O. P. Ferreira, Santos, and Souza 2024]

Let g be σ-strongly (geodesically) convex. Then

$$f(p^{(k+1)}) \leq f(p^{(k)}) - \frac{\sigma}{2} d^2(p^{(k)}, p^{(k+1)})$$

and $\displaystyle\sum_{k=0}^{\infty} d^2(p^{(k)}, p^{(k+1)}) < \infty$, so in particular $\displaystyle\lim_{k\to\infty} d(p^{(k)}, p^{(k+1)}) = 0$.

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A Numerical Example

The Difference of Convex Algorithm in Manopt.jl

The algorithm is implemented and released in Julia using `Manopt.jl`[3].
It can be used with any manifold from `Manifolds.jl`

A solver call looks like

```
q = difference_of_convex_algorithm(M, f, g, ∂h, p0)
```

where one has to implement `f(M, p)`, `g(M, p)`, and `∂h(M, p)`.

- a sub problem is generated if keyword `grad_g=` is set
- an efficient version of its cost and gradient is provided
- you can specify the sub-solver using `sub_state=`
  to also set up the specific parameters of your favourite algorithm

[3] see https://manoptjl.org/stable/solvers/difference_of_convex/

Rosenbrock and First Order Methods

Problem. We consider the classical Rosenbrock example[4]

$$\operatorname*{arg\,min}_{x\in\mathbb{R}^2} a\big(x_1^2 - x_2\big)^2 + \big(x_1 - b\big)^2,$$

where $a, b > 0$, usually $b = 1$ and $a \gg b$, here: $a = 2 \cdot 10^5$.

Known Minimizer $x^* = \begin{pmatrix} b \\ b^2 \end{pmatrix}$ with cost $f(x^*) = 0$.

Goal. Compare first-order methods, e. g. using the (Euclidean) gradient

$$\nabla f(x) = \begin{pmatrix} 4a(x_1^2 - x_2) \\ -2a(x_1^2 - x_2) \end{pmatrix} + \begin{pmatrix} 2(x_1 - b) \\ 0 \end{pmatrix}$$

[4] available online in ManoptExamples.jl

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A "Rosenbrock-Metric" on $\mathbb{R}^2$

In our Riemannian framework, we can introduce a new metric on $\mathbb{R}^2$ as

$$G_p := \begin{pmatrix} 1+4p_1^2 & -2p_1 \\ -2p_1 & 1 \end{pmatrix}, \text{ with inverse } G_p^{-1} = \begin{pmatrix} 1 & 2p_1 \\ 2p_1 & 1+4p_1^2 \end{pmatrix}.$$

We obtain $(X, Y)_p = X^{\mathrm{T}} G_p Y$

The exponential and logarithmic map are given as

$$\exp_p(X) = \begin{pmatrix} p_1 + X_1 \\ p_2 + X_2 + X_1^2 \end{pmatrix}, \qquad \log_p(q) = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 - (q_1 - p_1)^2 \end{pmatrix}.$$

`Manifolds.jl`:
Implement these functions on `MetricManifold(ℝ^2, RosenbrockMetric())`.

The Riemannian Gradient w.r.t. the new Metric

Let $f\colon \mathcal{M} \to \mathbb{R}$. Given the Euclidean gradient $\nabla f(p)$, its Riemannian gradient $\operatorname{grad} f\colon \mathcal{M} \to T\mathcal{M}$ is given by

$$\operatorname{grad} f(p) = G_p^{-1} \nabla f(p).$$

While we could implement this denoting $\nabla f(p) = \begin{pmatrix} f_1'(p) & f_2'(p) \end{pmatrix}^\mathrm{T}$ using

$$\left\langle \operatorname{grad} f(q), \log_q p \right\rangle_q = (p_1 - q_1) f_1'(q) + (p_2 - q_2 - (p_1 - q_1)^2) f_2'(q),$$

but it is automatically done in `Manopt.jl`.

The Experiment Setup

Algorithms. We now compare

1. The Euclidean gradient descent algorithm on $\mathbb{R}^2$,
2. The Riemannian gradient descent algorithm on $\mathcal{M}$,
3. The Difference of Convex Algorithm on $\mathbb{R}^2$,
4. The Difference of Convex Algorithm on $\mathcal{M}$.

For DCA third we split f into $f(x) = g(x) - h(x)$ with

$$g(x) = a\left(x_1^2 - x_2\right)^2 + 2\left(x_1 - b\right)^2 \quad \text{and} \quad h(x) = \left(x_1 - b\right)^2.$$

Initial point. $p_0 = \frac{1}{10}\begin{pmatrix}1\\2\end{pmatrix}$ with cost $f(p_0) \approx 7220.81$.

Stopping Criterion.
$d_{\mathcal{M}}(p^{(k)}, p^{(k-1)}) < 10^{-16}$ or $\lVert \operatorname{grad} f(p^{(k)})\rVert_p < 10^{-16}$.

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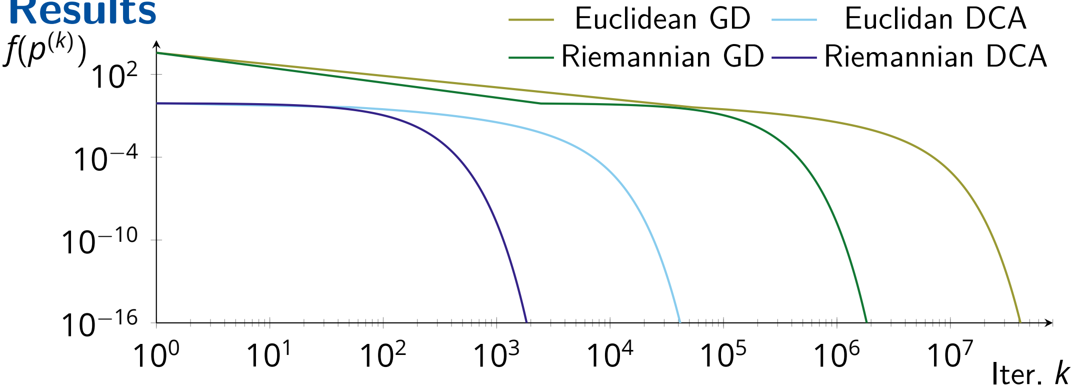
Results



Algorithm	**Runtime** (sec.)	**# Iterations**
Euclidean GD	305.567	53 073 227
Euclidean DCA	58.268	50 588
Riemannian GD	18.894	2 454 017
Riemannian DCA	7.704	2 459

Summary

- Introduced the Convex Bundle Method on manifolds to solve

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} f(p)$$

  - Provide an inner approximation of $\partial_\varepsilon f(p)$
  - A quadratic sub problem in a tangent space
  - Convergence of the Method on Hadamard manifolds
- Introduced the Difference of Convex Algorithm to solve

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} g(p) - h(p)$$

  - Relation to Fenchel Duality on Hadamard manifolds
  - Convergence on Hadamard manifolds

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