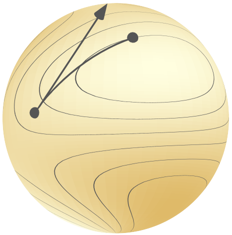

Manopt.jl

Optimization on Riemannian Manifolds

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Optimization

(Constrained) Optimization aims to find for a function $f\colon \mathbb{R}^m \to \mathbb{R}$ a point

$$\operatorname*{arg\,min}_{x\in\mathbb{R}^m} f(x)$$

Challenges:

- constrained to some $\mathcal{C} \subset \mathbb{R}^m$, e. g. unit vectors
- symmetries / invariances

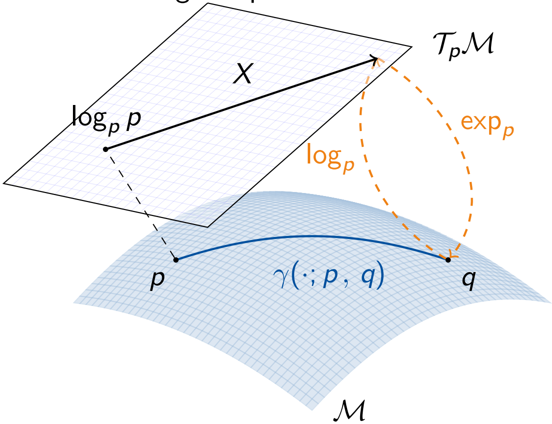
Geometric Optimization aims to find

$$\operatorname*{arg\,min}_{p\in\mathcal{M}} F(p)$$

where F is defined on a Riemannian manifold $\mathcal{M}$, e. g. the sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$.
$\Rightarrow$ the problem is unconstrained (again).

A Riemannian manifold $\mathcal{M}$

A d-dimensional Riemannian manifold can be informally defined as a set $\mathcal{M}$ covered with a 'suitable' collection of charts, that identify subsets of $\mathcal{M}$ with open subsets of $\mathbb{R}^d$ and a continuously varying inner product on the tangent spaces. [Absil, Mahony, and Sepulchre 2008]



Notation.

- Logarithmic map $\log_p q = \dot\gamma(0; p, q)$
- Exponential map $\exp_p X = \gamma_{p,X}(1)$
- Geodesic $\gamma(\cdot; p, q)$
- Tangent space $\mathcal{T}_p\mathcal{M}$
- inner product $(\cdot, \cdot)_p$

ManifoldsBase.jl & Manifolds.jl

`ManifoldsBase.jl` is an interface for Riemannian manifolds `M`

- `inner(M, p, X, Y)` $(X, Y)_p$
- `exp(M, p, X)` and `log(M, p, q)`,
- more general:
  `retract(M, p, X, m)`,
  where `m` is a retraction method
- embeddings as decorator
- ☺ mutating variants, e. g.
  `exp!(M, q, p, X)`
  works in place of `q`

🔗 juliamanifolds.github.io/ManifoldsBase.jl/
🔗 juliamanifolds.github.io/Manifolds.jl/

`Manifolds.jl` is a Library of manifolds

- Circle, (unit) Sphere & Torus
- Fixed Rank Matrices
- (Symplectic) Stiefel & Grassmann
- Hyperbolic space & Rotations
- Symmetric positive definite matrices
- ...and many more

as well as generically

- power & product manifold
- tangent & vector bundles
- Lie groups, connections, metrics,...

[Axen, Baran, RB, and Rzecki 2021]

▶ JuliaCon 2020 youtu.be/md-FnDGCh9M

Manopt.jl – Structure

`Manopt.jl` depends only on `ManifoldsBase.jl` and consists of

- a `Problem P` to specify static properties:
  the manifold $\mathcal{M}$, the cost F, its (Riemannian) gradient grad F, …
- some `Options O` to specify a solver and containing dynamic data:
  the current iterate, the current gradient, a stopping criterion, …
- implement
  1. `initialize_solver!(P, O)` to initialise a solver run
  2. `step_solver!(P, O, i)` to perform the ith step

⇒ call `solve(P,O)` to run the solver or use a high-level interface

Furthermore one can

- specify a `Stepsize s`, that is for example a `Linesearch l`
- include a `StoppingCriterion sc`, a functor `sc(P, O, i)` returning true/false

☺ `sc1 | sc2` and `sc1 & sc2` to build more advanced criteria

- `DebugOptions` and `RecordOptions` decorate Options with print/record

Manopt.jl – Available Solvers

Currently the following solvers are available

- Gradient Descent
  CG, Stochastic, Momentum, Alternating, Average, Nesterov, ...
- Quasi-Newton
  (L-)BFGS, DFP, Broyden, SR1, ...
- Nelder-Mead, Particle Swarm
- Subgradient Method
- Trust Regions
- Chambolle-Pock (PDHG)
- Douglas-Rachford
- Cyclic Proximal Point

The Manopt Family.

manoptjl.org [RB 2022]

manopt.org [Boumal, Mishra, Absil, and Sepulchre 2014]

pymanopt.org [Townsend, Koep, and Weichwald 2016]

Example – A Riemannian Center of Mass

The mean of N data points $x_1, \ldots, x_N \in \mathbb{R}^n$ is

$$x^* = \frac{1}{N}\sum_{i=1}^{N} x_i \quad \Leftrightarrow \quad x^* = \operatorname*{arg\,min}_{x \in \mathbb{R}^m} \frac{1}{2N}\sum_{i=1}^{N} \lVert x - x_i \rVert_2^2$$

$\Rightarrow$ the minimizer of sum of squared distances

For $p_1, \ldots, p_N \in \mathcal{M}$: [Karcher 1977]
Riemannian center(s) of mass are

$$\operatorname*{arg\,min}_{p \in \mathcal{M}} \frac{1}{2N}\sum_{i=1}^{N} d_{\mathcal{M}}^2(p, p_i),$$

- (in general) neither closed form nor unique
- For $F(p) = \frac{1}{2} d_{\mathcal{M}}^2(p, p_i)$ we have $\operatorname{grad} F(p) = -\log_p p_i$

$\Rightarrow$ use gradient descent

```
using LinearAlgebra
using Manopt, Manifolds
M = Sphere(2)
N = 100
pts = [randn(3) for _ in 1:N]
pts ./= norm.(pts)
F(M, p) = sum(
  pi -> distance(M,pi,p)^2/2N,
  pts,
)
gF(M, p) = sum(
  pi -> grad_distance(M,pi,p)/N,
  pts,
)
# compute a center of mass
# in place of m
m = copy(M, pts[1])
gradient_descent!(M, F, gF, m)
```

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