A Graph Framework for Manifold-valued Data

Ronny Bergmann[a]
University of Kaiserslautern

S21 Mathematical Signal and Image Processing, GAMM 2017,

Weimar, March 7, 2017

[a]joint work with D. Tenbrinck (WWU Münster)

Contents

Introduction

History, kind of...

Back then, September '15 in Münster...

The graph

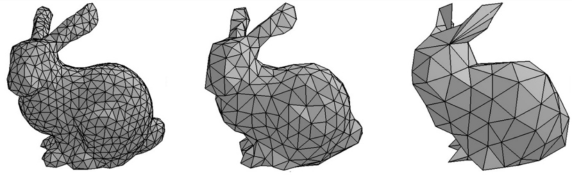
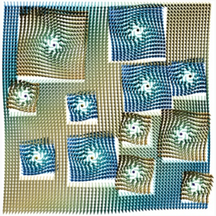


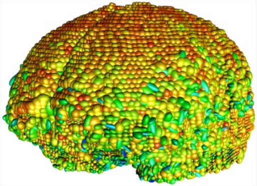
Image credit: G. Peyre

Same setting as in the last talk by Daniel:
Data given on a directed graph $G = (V, E, w)$ with

- a finite set of nodes V
- a finite set of directed edges $E \subset V \times V$
- a (symmetric) weight function $w : V \times V \to \mathbb{R}^+$,
  $w(u, v) = 0$ for $v \not\sim u$.

Manifold-valued functions

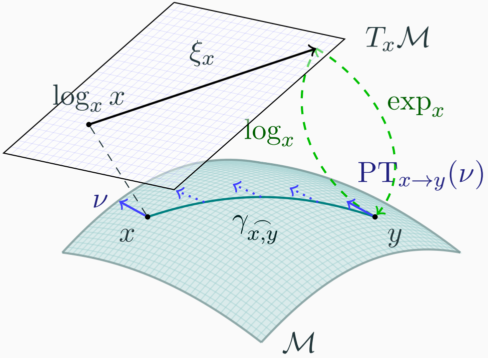




Our data $f\colon V \to \mathcal{M}$ is manifold-valued, e. g.,

$\mathbb{S}^1$	InSAR, HSI(HSV) color space, phase space
$\mathbb{S}^2$	directions, chromaticity-brightness colorspace
$\mathrm{SO}(3)/\mathcal{S}$	orientations, electron backscattered diffraction
$\mathcal{P}(s)$	DT-MRI, covariance matrices

Notations on a Riemannian manifold $\mathcal{M}$



geodesic $\gamma_{\widehat{x,y}}$ shortest path (on $\mathcal{M}$) connecting $x, y \in \mathcal{M}$.
tangential plane $\mathrm{T}_x\mathcal{M}$ at x, $\mathrm{T}\mathcal{M} := \cup_{x\in\mathcal{M}}\mathrm{T}_x\mathcal{M}$
logarithmic map $\log_x y = \dot{\gamma}_{\widehat{x,y}}(0)$, "velocity towards y"
exponential map $\exp_x \xi_x = \gamma(1)$, where $\gamma(0) = x$, $\dot{\gamma}(0) = \xi_x$
parallel transport $\mathrm{PT}_{x\to y}(\nu)$ of $\nu \in \mathrm{T}_x\mathcal{M}$ along $\gamma_{\widehat{x,y}}$

The Graph Framework

The basic idea

Real-valued case

$$\mathcal{H}(V;\mathbb{R}^m) = \{f\colon V \to \mathbb{R}^m\}$$

Manifold-valued case

$$\mathcal{H}(V;\mathcal{M}) \coloneqq \{f\colon V \to \mathcal{M}\}$$

Space of edge functions

$$\mathcal{H}(E;\mathbb{R}^m) = \{H\colon E \to \mathbb{R}^m,\ H(u,v) \in \mathbb{R}^m, (u,v) \in E\}$$

$$\mathcal{H}(E;T_f\mathcal{M}) = \{H_f\colon E \to \mathrm{T}\mathcal{M},\ H_f(u,v) \to \mathrm{T}_{f(u)}\mathcal{M}, (u,v) \in E\}$$

Gradient

$$\nabla f(u,v) = \sqrt{w(u,v)}\big(f(v) - f(u)\big)$$

$$\nabla f(u,v) \coloneqq \sqrt{w(u,v)}\log_{f(u)} f(v) \in T_{f(u)}\mathcal{M}$$

Local variation

$$\|\nabla f\|^p_{p,f(u)} = \sum_{v\sim u} \sqrt{w(u,v)}^p \, \|f(v)-f(u)\|^p$$

$$\|\nabla f\|^p_{p,f(u)} \coloneqq \sum_{v\sim u} \sqrt{w(u,v)}^p d_{\mathcal{M}}(f(u),f(v))^p$$

(Local) divergence

What is $\langle \nabla f, H \rangle = \langle f, \nabla^\star H \rangle$, $\nabla^\star = -\operatorname{div}$, on a manifold?

(Local) divergence

Theorem [RB, Tenbrinck, 2017]

For $f \in \mathcal{H}(V;\mathcal{M})$, $H_f \in \mathcal{H}(E;\mathrm{T}_f\mathcal{M})$, we have

$$\langle \nabla f, H_f \rangle_{\mathcal{H}(E;\mathrm{T}_f\mathcal{M})} = \sum_{u\in V}\sum_{v\sim u} \langle \log_{f(u)} f(v), -\operatorname{div} H_f(u)\rangle_{f(u)},$$

where the local divergence is given by

$$\operatorname{div} H_f(u) := \frac{1}{2}\sum_{v\sim u} \sqrt{w(v,u)}\,\mathrm{PT}_{f(v)\to f(u)} H_f(v,u) - \sqrt{w(u,v)} H_f(u,v)$$

Remark

By antisymmetry $\nabla f(u,v) = -\mathrm{PT}_{f(v)\to f(u)} \nabla f(v,u) \in \mathrm{T}_{f(u)}\mathcal{M}$ we get

$$\operatorname{div}(\nabla f)(u) = -\sum_{v\sim u} w(u,v) \log_{f(u)} f(v)$$

The manifold-valued graph p-Laplacians

We define the p-Graph-Laplacians:

- anisotropic $\Delta_p^{\mathrm{a}} \colon \mathcal{H}(V;\mathcal{M}) \to \mathcal{H}(V;T\mathcal{M})$ by

$$\begin{aligned}\Delta_p^{\mathrm{a}} f(u) &:= \operatorname{div}\bigl(\lVert\nabla f\rVert_{f(\cdot)}^{p-2}\,\nabla f\bigr)(u)\\ &= -\sum_{v\sim u}\sqrt{w(u,v)}^{p}\,d_{\mathcal{M}}^{p-2}(f(u),f(v))\log_{f(u)} f(v)\end{aligned}$$

- isotropic $\Delta_p^{\mathrm{i}} \colon \mathcal{H}(V;\mathcal{M}) \to \mathcal{H}(V;T_f\mathcal{M})$ by

$$\begin{aligned}\Delta_p^{\mathrm{i}} f(u) &:= \operatorname{div}\bigl(\lVert\nabla f\rVert_{2,f(\cdot)}^{p-2}\nabla f\bigr)(u)\\ &= -\,b_{\mathrm{i}}(u)\sum_{v\sim u} w(u,v)\log_{f(u)} f(v)\ ,\end{aligned}$$

  where

$$b_{\mathrm{i}}(u) := \lVert\nabla f\rVert_{2,f(u)}^{p-2} = \Bigl(\sum_{v\sim u} w(u,v)\,d_{\mathcal{M}}^{2}(f(u),f(v))\Bigr)^{\frac{p-2}{2}}.$$

Variational optimization problems

Goal: A Minimizer of a Variational Model $\mathcal{E} \colon \mathcal{H}(V;\mathcal{M}) \to \mathbb{R}$

the anisotropic energy functional

[Lellmann, Strekalovskiy, Kötters, Cremers, '13; Weinmann, Demaret, Storath, '14; RB, Persch, Steidl, '16]

$$\mathcal{E}_{\mathrm{a}}(f) \coloneqq \frac{\lambda}{2} \sum_{u \in V} d^2_{\mathcal{M}}(f_0(u), f(u)) + \frac{1}{p} \sum_{(u,v) \in E} \|\nabla f(u,v)\|^p_{f(u)},$$

and the isotropic energy functional

[RB, Chan, Hielscher, Persch, Steidl, '16]

$$\mathcal{E}_{\mathrm{i}}(f) \coloneqq \frac{\lambda}{2} \sum_{u \in V} d^2_{\mathcal{M}}(f_0(u), f(u)) + \frac{1}{p} \sum_{u \in V} \Bigl(\sum_{v \sim u} \|\nabla f(u,v)\|^2_{f(u)} \Bigr)^{p/2}.$$

Optimality conditions

For $e \in \{\mathrm{a}, \mathrm{i}\}$ and any $u \in V$ we have for a minimizer

$$0 \overset{!}{=} \Delta_p^e f(u) - \lambda \log_{f(u)} f_0(u) \in \mathrm{T}_{f(u)}\mathcal{M}.$$

Algorithm I. Forward difference or explicit scheme:

$$f_{n+1}(u) = \exp_{f_n(u)}\bigl(\Delta t\,(\Delta_p^{\mathrm{e}} f_n(u) - \lambda \log_{f_n(u)} f_0(u))\bigr)$$

! to meet CFL conditions: small Δt necessary

Algorithm II. Jacobi iteration

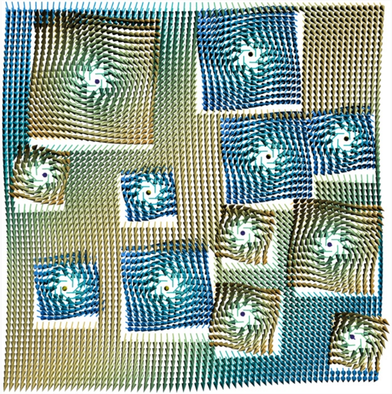
$$f_{n+1}(u) = \exp_{f_n(u)}\left(\frac{\sum_{v\sim u} b(u,v) \log_{f_n(u)} f_n(v) + \lambda \log_{f_n(u)} f_0(u)}{\lambda + \sum_{v\sim u} b(u,v)}\right),$$

$$b(u,v) = \begin{cases} \sqrt{w(u,v)}^p d_{\mathcal{M}}^{p-2}(f(u), f(v)), & e = \mathrm{a},\\ b_{\mathrm{i}}(u), & e = \mathrm{i}. \end{cases}$$

Numerical Examples

Evolution for the graph p-Laplacian

- $\mathcal{M} = \mathbb{S}^2$
- $V = \{1, \dots, 64\} \times \{1, \dots, 64\}$ pixel grid
- E is the 4-neighborhood, Neumann boundary

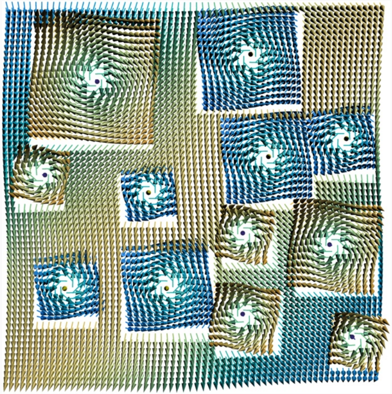


f_0

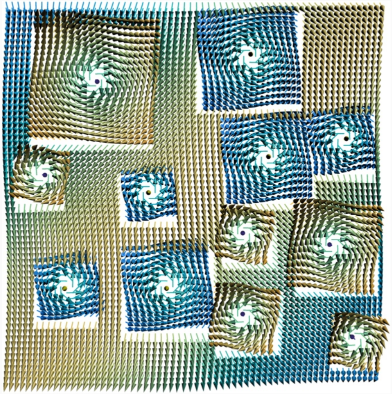
$\lambda = 0$ (no data term)

Evolution for the graph p-Laplacian

- $\mathcal{M} = \mathbb{S}^2$
- $V = \{1, \ldots, 64\} \times \{1, \ldots, 64\}$ pixel grid
- E is the 4-neighborhood, Neumann boundary



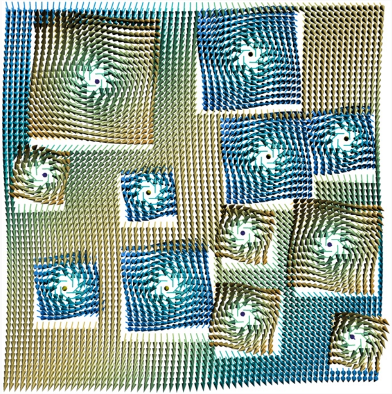
f_0



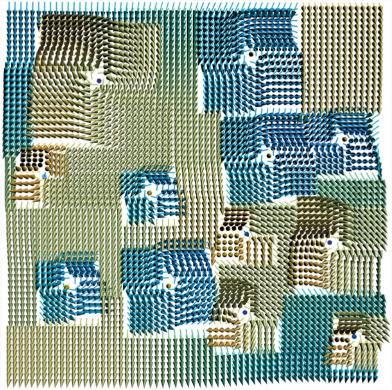
$\lambda = 0$, $p = 1$, anisotrop

Evolution for the graph p-Laplacian

- $\mathcal{M} = \mathbb{S}^2$
- $V = \{1, \ldots, 64\} \times \{1, \ldots, 64\}$ pixel grid
- E is the 4-neighborhood, Neumann boundary



f_0

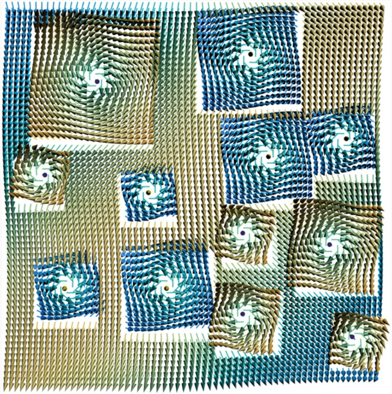


f_{1000}

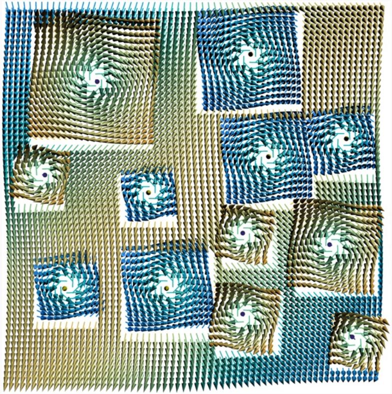
$\lambda = 0$, $p = 1$, anisotrop

Evolution for the graph p-Laplacian

- $\mathcal{M} = \mathbb{S}^2$
- $V = \{1, \ldots, 64\} \times \{1, \ldots, 64\}$ pixel grid
- E is the 4-neighborhood, Neumann boundary



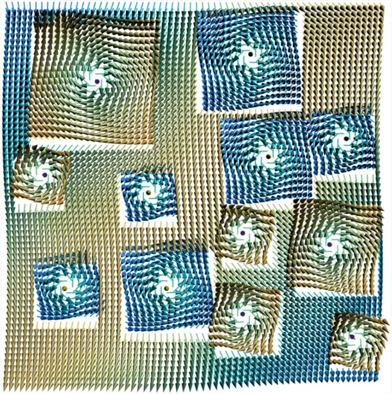
f_0



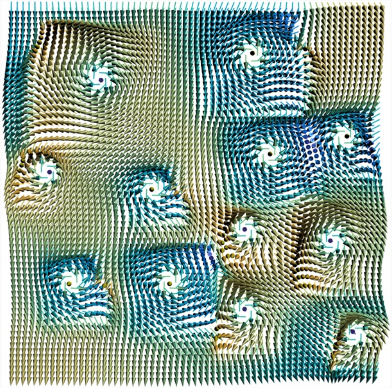
$\lambda = 0$, $p = 1$, isotrop

Evolution for the graph p-Laplacian

- $\mathcal{M} = \mathbb{S}^2$
- $V = \{1, \dots, 64\} \times \{1, \dots, 64\}$ pixel grid
- E is the 4-neighborhood, Neumann boundary



f_0

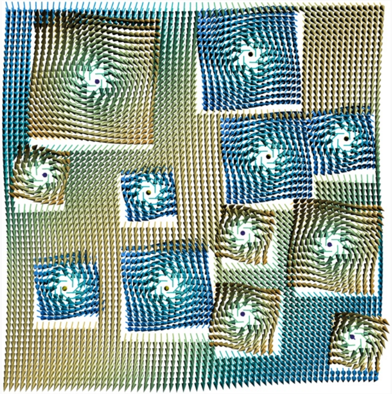


f_{1000}

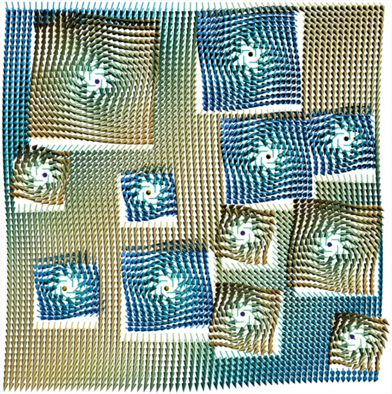
$\lambda = 0$, $p = 1$, isotrop

Evolution for the graph p-Laplacian

- $\mathcal{M} = \mathbb{S}^2$
- $V = \{1, \ldots, 64\} \times \{1, \ldots, 64\}$ pixel grid
- E is the 4-neighborhood, Neumann boundary



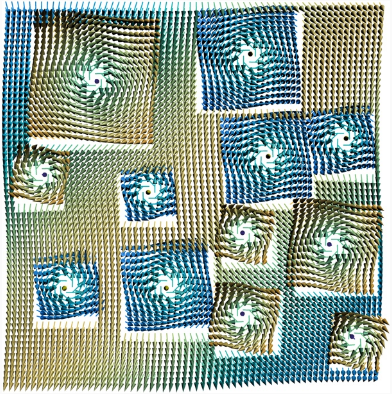
f_0



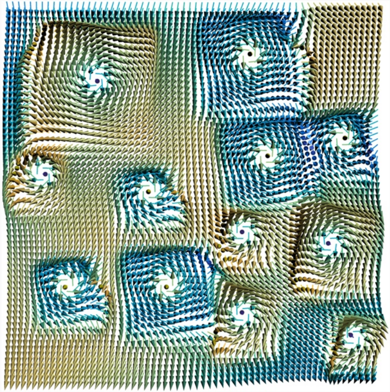
$\lambda = 0$, $p = 2$, (an)isotrop

Evolution for the graph p-Laplacian

- $\mathcal{M} = \mathbb{S}^2$
- $V = \{1, \ldots, 64\} \times \{1, \ldots, 64\}$ pixel grid
- E is the 4-neighborhood, Neumann boundary



f_0

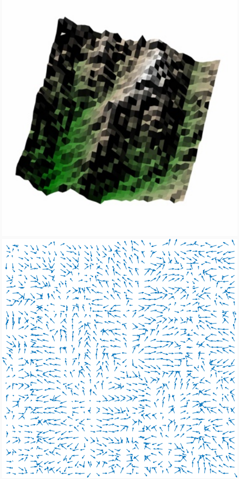


f_{1000}

$\lambda = 0$, $p = 2$, (an)isotrop

Local image denoising

Light Detection and Ranging data (LiDaR), 40 × 40 pixel



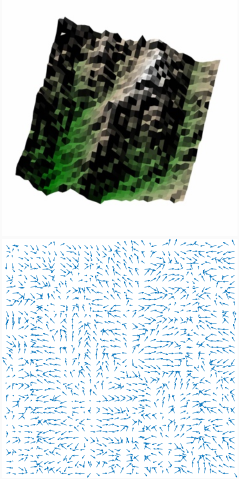
original data

[Geesch et al., 2009] via MFOPT

lellmann.net/software/mfopt

Local image denoising

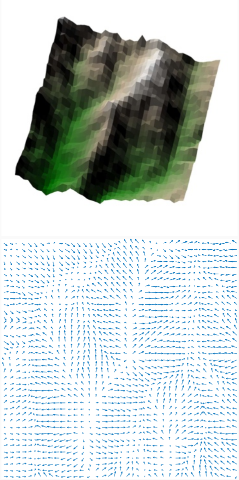
Light Detection and Ranging data (LiDaR), 40×40 pixel



original data

[Geesch et al., 2009] via MFOPT

lellmann.net/software/mfopt

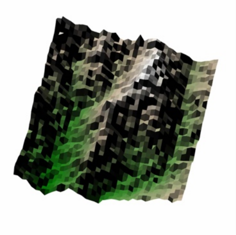


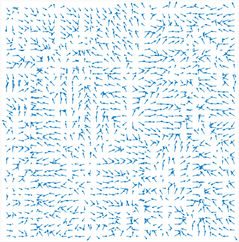
$p = 2$, $\lambda = 0.5$,

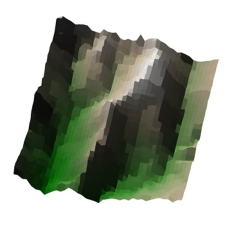
(an)isotrop.

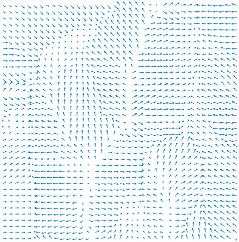
Local image denoising

Light Detection and Ranging data (LiDaR), 40×40 pixel







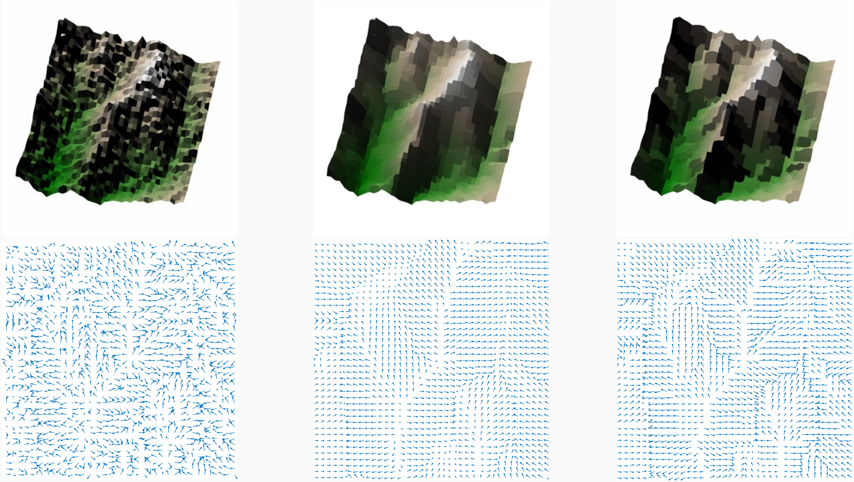


original data
[Geesch et al., 2009] via MFOPT
lellmann.net/software/mfopt

$p = 1$, $\lambda = 2$,
anisotrop.

Local image denoising

Light Detection and Ranging data (LiDaR), 40×40 pixel



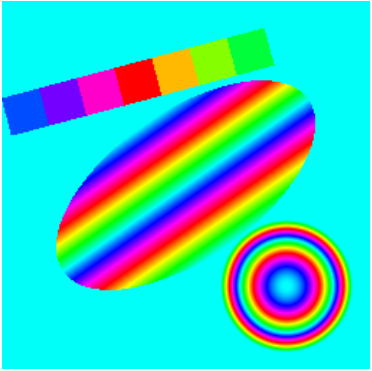
original data
[Geesch et al., 2009] via MFOPT
lellmann.net/software/mfopt

$p = 1$, $\lambda = 2$,
anisotrop.

$p = 0.1$, $\lambda = 1$,
anisotrop.

Nonlocal image denoising

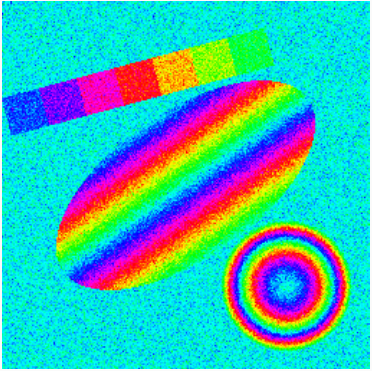
- $\mathcal{M} = \mathbb{S}^1$, phase in $[-\pi, \pi)$, color: hue
- $V = \{1, \ldots, 256\} \times \{1, \ldots, 256\}$ pixel grid
- E from 12 most similar pixels w.r.t. 17×17 patch distances



original.

Nonlocal image denoising

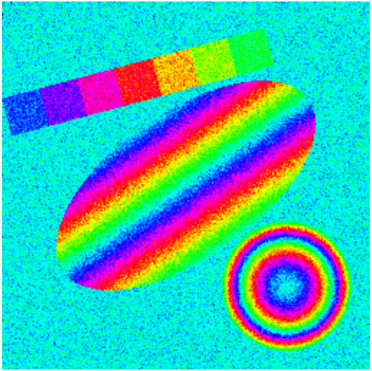
- $\mathcal{M} = \mathbb{S}^1$, phase in $[-\pi, \pi)$, color: hue
- $V = \{1, \ldots, 256\} \times \{1, \ldots, 256\}$ pixel grid
- E from 12 most similar pixels w.r.t. 17×17 patch distances



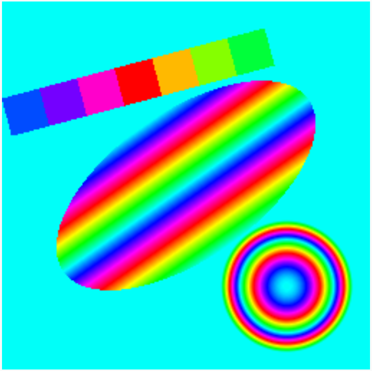
wrapped Gaussian, $\sigma = 0.3$.

Nonlocal image denoising

- $\mathcal{M} = \mathbb{S}^1$, phase in $[-\pi, \pi)$, color: hue
- $V = \{1, \ldots, 256\} \times \{1, \ldots, 256\}$ pixel grid
- E from 12 most similar pixels w.r.t. 17×17 patch distances



wrapped Gaussian, $\sigma = 0.3$.

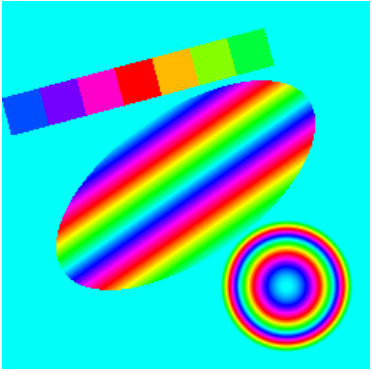


NL-MSSE. [Laus et al., 2017]

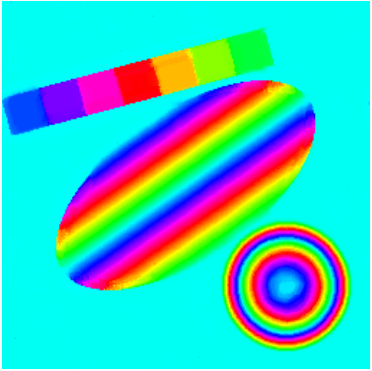
$\varepsilon = 2.50 \times 10^{-3}$

Nonlocal image denoising

- $\mathcal{M} = \mathbb{S}^1$, phase in $[-\pi, \pi)$, color: hue
- $V = \{1, \ldots, 256\} \times \{1, \ldots, 256\}$ pixel grid
- E from 12 most similar pixels w.r.t. 17×17 patch distances



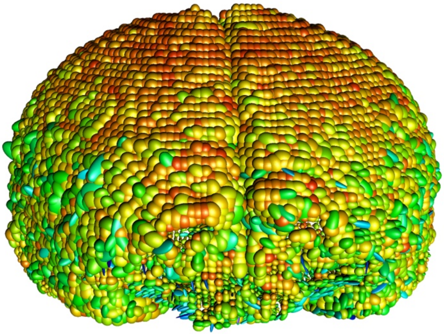
original.



anisotropic, $p = 1$, $\lambda = 2^{-8}$, $\varepsilon = 2.67 \times 10^{-3}$.

Local denoising on a surface

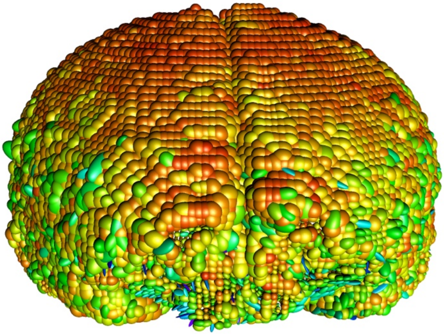
- $\mathcal{M} = \mathcal{P}(3)$
- V = point cloud: boundary of Camino dataset[1]
- local Neighborhood, $d_{\max} = 2$



[1] Data available from The Camino Project, cmic.cs.ucl.ac.uk/camino

Local denoising on a surface

- $\mathcal{M} = \mathcal{P}(3)$
- V = point cloud: boundary of Camino dataset[1]
- local Neighborhood, $d_{\max} = 2$



$\lambda = 50$, anisotropic 1-Laplace.

[1] Data available from The Camino Project, cmic.cs.ucl.ac.uk/camino

Conclusion

Conclusion

- graph p-Laplace to manifold valued vertex functions
- local divergence
- explicit & Jacobi type algorithms to solve corresponding variational functionals
- unified framework for local and nonlocal methods

Conclusion

- graph p-Laplace to manifold valued vertex functions
- local divergence
- explicit & Jacobi type algorithms to solve corresponding variational functionals
- unified framework for local and nonlocal methods

Future work

- other graph based PDEs (Eikonal, ∞-Laplacian)
- other image processing tasks (segmentation, inpainting)
- other numerical schemes

- RB and D. Tenbrinck. *A Graph Framework for manifold-valued Data*. 2017. arXiv: 1702.05293.
- RB, J. Persch, and G. Steidl. "A Parallel Douglas–Rachford Algorithm for Minimizing ROF-like Functionals on Images with Values in Symmetric Hadamard Manifolds". In: *SIAM J. Imag. Sci.* 9.4 (2016), pp. 901–937.
- RB, R. H. Chan, R. Hielscher, J. Persch, and G. Steidl. "Restoration of Manifold-Valued Images by Half-Quadratic Minimization". In: *Inv. Probl. Imag.* 2.10 (2016), pp. 281–304.
- F. Laus, M. Nikolova, J. Persch, and G. Steidl. "A Nonlocal Denoising Algorithm for Manifold-Valued Images Using Second Order Statistics". In: *SIAM J. Imag. Sci.* (2017). accepted.
- A. Elmoataz, M. Toutain, and D. Tenbrinck. "On the p-Laplacian and ∞-Laplacian on Graphs with Applications in Image and Data Processing". In: *SIAM J. Imag. Sci.* 8.4 (2015), pp. 2412–2451.

Thank you for your attention.

RB and D. Tenbrinck. *A Graph Framework for manifold-valued Data*. 2017. arXiv: 1702.05293.

RB, J. Persch, and G. Steidl. “A Parallel Douglas–Rachford Algorithm for Minimizing ROF-like Functionals on Images with Values in Symmetric Hadamard Manifolds”. In: *SIAM J. Imag. Sci.* 9.4 (2016), pp. 901–937.

RB, R. H. Chan, R. Hielscher, J. Persch, and G. Steidl. “Restoration of Manifold-Valued Images by Half-Quadratic Minimization”. In: *Inv. Probl. Imag.* 2.10 (2016), pp. 281–304.

F. Laus, M. Nikolova, J. Persch, and G. Steidl. “A Nonlocal Denoising Algorithm for Manifold-Valued Images Using Second Order Statistics”. In: *SIAM J. Imag. Sci.* (2017). accepted.

A. Elmoataz, M. Toutain, and D. Tenbrinck. “On the p-Laplacian and ∞-Laplacian on Graphs with Applications in Image and Data Processing”. In: *SIAM J. Imag. Sci.* 8.4 (2015), pp. 2412–2451.