Technische Universität Kaiserslautern

Inpainting of Cyclic Data Using First and Second Order Differences

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Contents

Introduction

- employing the Rudin-Osher-Fatemi (ROF) functional [Osher, Rudin, Fatemi, 1992]

$$\sum_{i,j}(f_{i,j} - x_{i,j})^2 + \lambda \sum_{i,j} |\nabla x_{i,j}|$$

  - f noisy image
  - ∇ discrete gradient
  - $\sum_{i,j} |\nabla x_{i,j}|$ discrete total variation (TV)
  - regularization parameter $\lambda > 0$

$\Rightarrow$ edge-preserving

- several higher order variational models avoid stair caising-effect
[Chambolle, Lions, 1997; Setzer, Steidl, 2008; Bredies, Kunisch, Pock, 2010; Papafitsoros, Schönlieb, 2014]

Recently [Strekalovski, Cremers, 2013], [Lellmann et al., 2013], [Weinmann et al., 2013], [Bačák, 2013]

- TV denoising generalized to Riemannian manifolds
- several algorithms to find a minimizer x^*, e.g. PPA
- convergence for PPA on CAT(0) spaces (does not include $\mathbb{S}^1$)

Finite Differences on $\mathbb{R}$

Let $w = (w_j)_{j=1}^d \in \mathbb{R}^d \setminus \{0\}$ fulfill

$$\langle w, 1_d \rangle := \sum_{j=1}^{d} w_j = 0.$$

The finite difference operator is given by

$$\Delta(x; w) := \langle x, w \rangle, \quad x \in \mathbb{R}^d.$$

This talk: $w \in \{b_1, b_2, b_{1,1}\}$

- $b_1 = (-1, 1)$ for f_x, f_y
- $b_2 = (1, -2, 1)$ for f_{xx}, f_{yy}
- $b_{1,1} = (1, -1, -1, 1)$ for f_{xy}

Finite Differences on the Circle $\mathbb{S}^1$

Cyclic data

$$p_i \in \mathbb{S}^1 := \{q \in \mathbb{R}^2 \,:\, \|q\|_2 = 1\} \Longleftrightarrow x_i \in [-\pi, \pi)$$

Distances

- $d(x; b_1) := \arccos\langle p_1, p_2\rangle = |(x_2 - x_1)_{2\pi}| = |(\langle x, b_1\rangle)_{2\pi}|$ (arc length)

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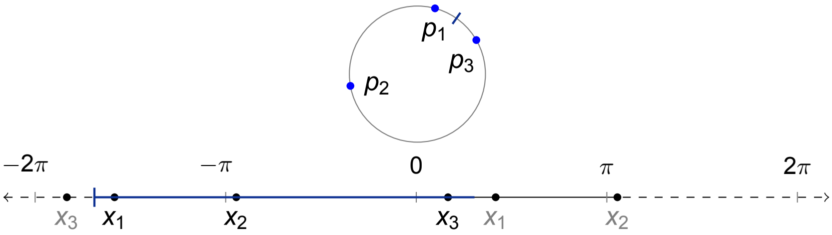
- $d(x; b_2)$? Several unwrappings to compute $x_1 - 2x_2 + x_3$



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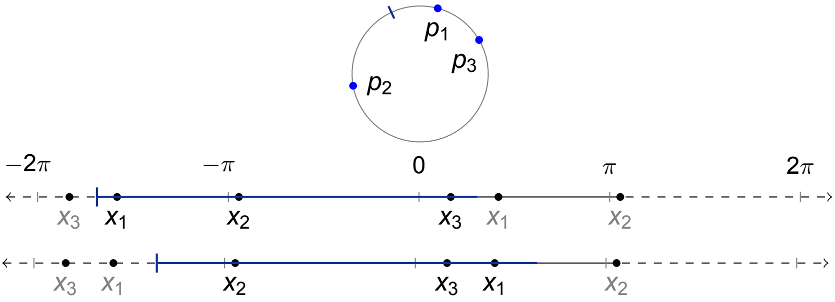

Finite Differences on the Circle $\mathbb{S}^1$

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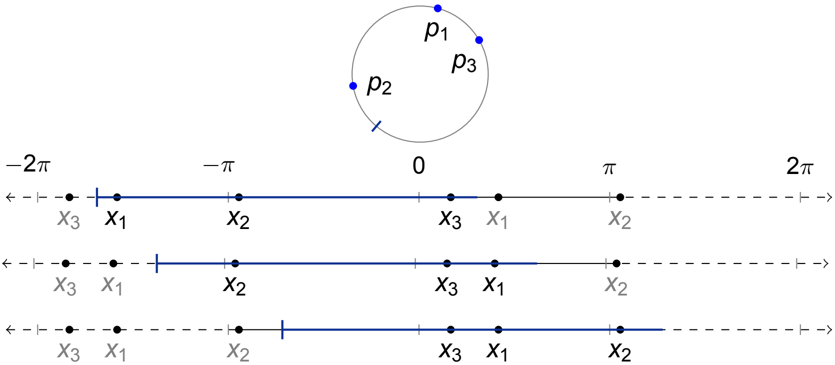

Absolute Finite Differences of Cyclic Data

The cyclic absolute finite difference w.r.t. w

$$d(x; w) := \min_{\mu\in\mathbb{R}} \left|\Delta\big([x + \mu 1_d]_{2\pi}; w\big)\right|, \quad x \in [-\pi, \pi)^d$$

- $[x]_{2\pi}$: element-wise mod 2π, except for $x_i = (2k + 1)\pi$: take both $\pm\pi$
- $d(x; w)$ is shift invariant
- notation: $d_1 := d(\cdot\,; b_1)$, $d_2 := d(\cdot\,; b_2)$ and $d_{1,1} := d(\cdot\,; b_{1,1})$
- minimization not necessary for $w \in \{b_1, b_2, b_{1,1}\}$: it holds

$$d(x; w) = \left|(\Delta(x, w))_{2\pi}\right|$$

this does not hold e.g. for $w = b_3 = (1, -3, 3, -1)$

Second Order TV for Cyclic Data

For given data $f = (f_i)_{i=1}^N \in (\mathbb{S}^1)^N$, $\alpha, \beta \geq 0$, compute

$$\operatorname*{arg\,min}_{x\in[-\pi,\pi)^N} J(x), \quad J(x) := \sum_{i=1}^{N} d_1(f_i, x_i)^2 + \alpha\,\mathrm{TV}_1(x) + \beta\,\mathrm{TV}_2(x)$$

where

$$\mathrm{TV}_1(x) = \sum_{i=1}^{N-1} d_1(x_i, x_{i+1}), \quad \mathrm{TV}_2(x) = \sum_{i=2}^{N-1} d_2(x_{i-1}, x_i, x_{i+1}).$$

Similar: 2D model for data $f_{i,j} \in (\mathbb{S}^1)^{N,M}$

- vertical and horizontal first and second order differences
- mixed second order difference on each 2×2 submatrix of $x \in [-\pi, \pi)^{N,M} \Rightarrow$ additional term $\gamma\,\mathrm{TV}_{1,1}(x)$

Models for Inpainting

- image domain: $\Omega_0 \subset \mathbb{N}^2$ with $\Omega \subset \Omega_0$ unknown
- data $f_{i,j}$, $(i,j) \in \overline{\Omega} = \Omega_0 \backslash \Omega$
- noiseless case

$$\operatorname*{arg\,min}_{x \in [-\pi,\pi)^{N,M}} \alpha \operatorname{TV}_1^\Omega(x) + \beta \operatorname{TV}_2^\Omega(x) + \gamma \operatorname{TV}_{1,1}^\Omega(x),$$
$$\text{subject to} \quad x_{i,j} = f_{i,j} \quad \text{for all} \quad (i,j) \in \overline{\Omega}$$

- noisy case

$$\operatorname*{arg\,min}_{x \in [-\pi,\pi)^{N,M}} \sum_{(i,j) \in \overline{\Omega}} d_1(f_{i,j}, x_{i,j})^2 + \alpha \operatorname{TV}_1(x) + \beta \operatorname{TV}_2(x) + \gamma \operatorname{TV}_{1,1}(x)$$

- iterative initialization: set $x_{i,j}$, $(i,j) \in \Omega$, by solving $d(x;w) = 0$, whenever all other data items are known [Almeida, Figueiredo, 2013]

$\Rightarrow$ "cold start initialization"

Proximal Mappings on $\mathbb{S}^1$

[Rockafellar, 1976; Ferreira, Oliveira, 2002]

$$\operatorname{prox}_{\lambda\varphi}(g) = \operatorname*{arg\,min}_{x\in[-\pi,\pi)^d} \frac{1}{2}\sum_{i=1}^{d} d_1(g_i, x_i)^2 + \lambda\varphi(x), \quad \lambda > 0$$

Theorem: Proximal Mapping I [B., Laus, Steidl, Weinmann, 2014]

The unique minimizer x^* of $\operatorname{prox}_{\lambda d_1(f,\,\cdot\,)^2}(g)$ is

$$x^* = \left(\frac{g+\lambda f}{1+\lambda} + \frac{\lambda}{1+\lambda} 2\pi\, v\right)_{2\pi}, \quad v = \begin{cases} 0 & \text{for } |g-f| \leq \pi, \\ \operatorname{sgn}(g-f) & \text{for } |g-f| > \pi. \end{cases}$$

Proximal Mappings on $\mathbb{S}^1$

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Theorem: Proximal Mapping II [B., Laus, Steidl, Weinmann, 2014]

The minimizers of $\operatorname{prox}_{\lambda d(\cdot;w)}(g)$, $w \in \{b_1, b_2, b_{1,1}\}$, are given by

$$x^* = \left(g - \operatorname{sgn}([\langle g,w\rangle]_{2\pi}) \min\left\{\lambda, \frac{|(\langle g,w\rangle)_{2\pi}|}{\|w\|_2^2}\right\} w\right)_{2\pi}.$$

For $|(\langle g,w\rangle)_{2\pi}| = \pi$ there are two minimizers, otherwise it is unique.

The Cyclic Proximal Point Algorithm

Find $\arg\min_x \varphi(x)$, $\varphi\colon \mathbb{R}^N \to \mathbb{R}$, convex, proper, lsc, by Picard iteration:

[Moreau, 1965; Rockafellar, 1976]

$$x^{(k)} = \operatorname{prox}_{\lambda\varphi}(x^{(k-1)}), \quad k > 0$$

$\Rightarrow$ fast evaluation of $\operatorname{prox}_{\lambda\varphi}$ needed

For $\varphi = \sum_{i=1}^{c} \varphi_i$ use Cyclic Proximal Point Algorithm (CPPA) [Bertsekas, 2011]

$$x^{(k+\frac{i+1}{c})} = \operatorname{prox}_{\lambda_k\varphi_i}(x^{(k+\frac{i}{c})}), \quad i = 0,\ldots,c-1,\ k > 0$$

converges to a minimizer if $\{\lambda_k\} \in \ell_2(\mathbb{Z})\setminus\ell_1(\mathbb{Z})$.

For our model $J\colon (\mathbb{S}^1)^N \to \mathbb{R}$ we can prove convergence if additionally

- data f_i *locally* dense enough
- $\alpha, \beta(,\gamma)$ are sufficiently small

An Efficient Splitting for the CPPA

- data $\displaystyle \frac{1}{2}\sum_{i=1}^{N} d_1(f_i, x_i)^2 =: J_1(x)$

  proximal mapping I (simultaneously elementwise)

- first order differences

$$\alpha \,\mathrm{TV}_1(x) = \alpha \sum_{i=1}^{N-1} d_1(x_i, x_{i+1})$$

- second order differences

$$\beta \,\mathrm{TV}_2(x) = \beta \sum_{i=2}^{N-1} d_2(x_{i-1}, x_i, x_{i+1})$$

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$$\alpha\,\mathrm{TV}_1(x) = \sum_{l=0}^{1} \alpha \sum_{i=1}^{\lfloor \frac{N-1}{2} \rfloor} d_1(x_{2i-1+l}, x_{2i-l}) =: \sum_{l=0}^{1} J_{2+l}(x)$$

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  inner sum: distinct data $\Rightarrow$ proximal mapping II with $w = b_1$
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An Efficient Splitting for the CPPA

- data $\frac{1}{2}\sum_{i=1}^{N} d_1(f_i, x_i)^2 =: J_1(x)$

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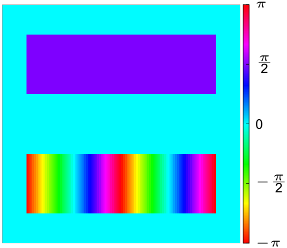
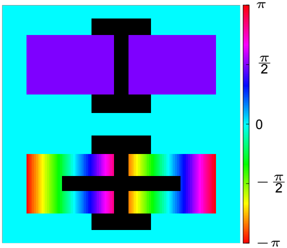
$\Rightarrow$ $J(x) = \sum_{l=1}^{6} J_l(x)$ $\Rightarrow$ cycle length $c = 6$ (2D: $c = 15$)

Example: Inpainting



Original image.

Example: Inpainting



Original image (lost area in black).

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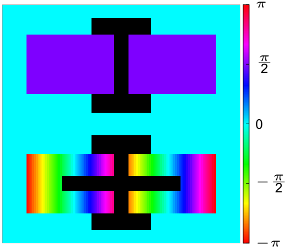

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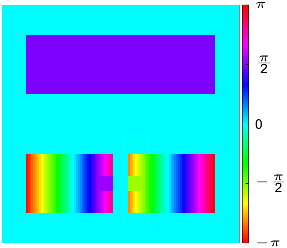
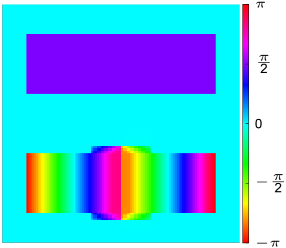
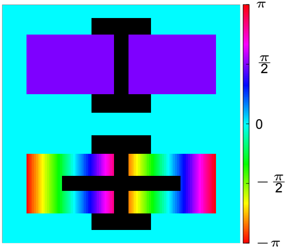


First order real-valued model.



Our model, $\alpha = 2$, $\beta = \gamma = 0$.

Example: Inpainting



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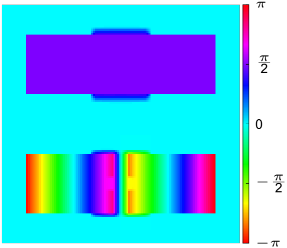
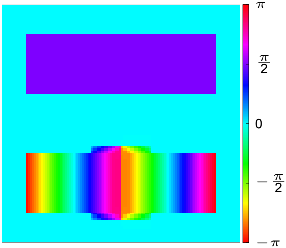


First & second order real-valued model.



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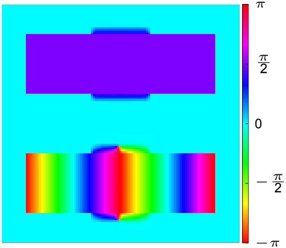
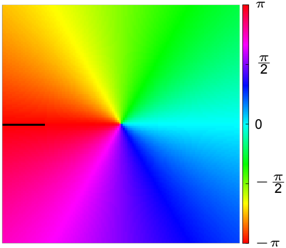
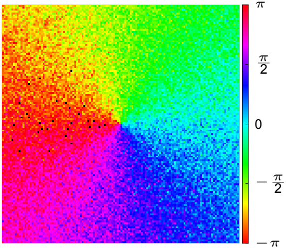


Our model, $\alpha = 2$, $\beta = \gamma = 1$.

Example: Combined Inpainting and Denoising



Original image.



Noisy image, $\sigma = 0.3$.

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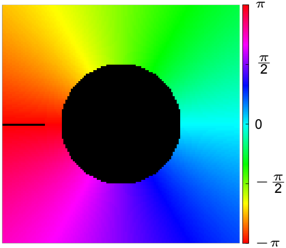

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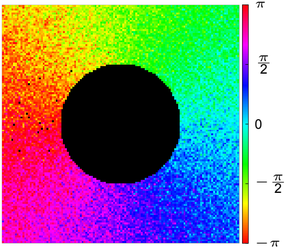

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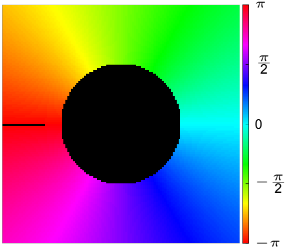

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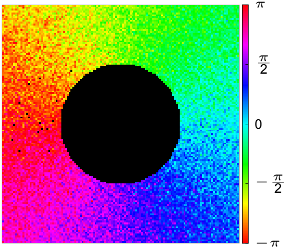

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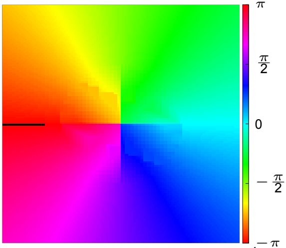

Noiseless model, $\alpha = \frac{1}{8}$.

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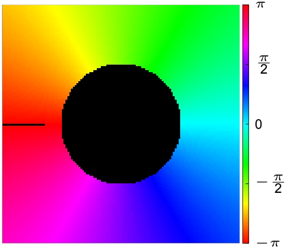

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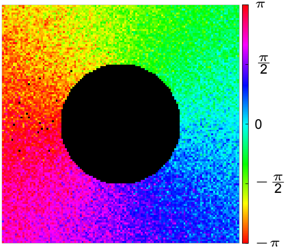
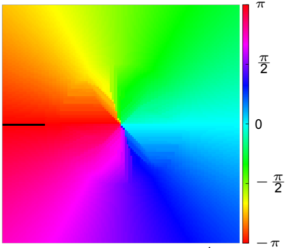
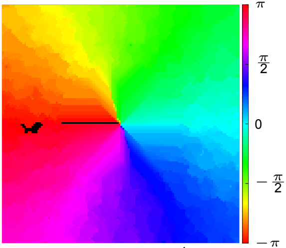


Noisy image, $\sigma = 0.3$.



Noiseless model, $\alpha = \frac{1}{8}$ (& diags).



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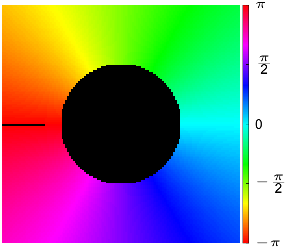

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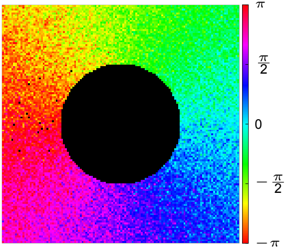

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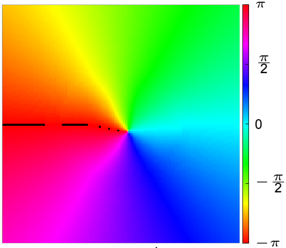
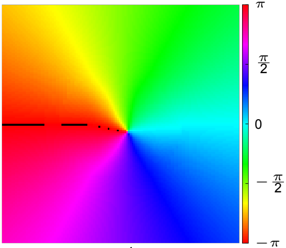


Noiseless model, $\alpha = \frac{1}{8}$ (& diags), $\beta = \gamma = \frac{1}{4}$.



Noisy model, $\alpha = \frac{1}{8}$ (& diags), $\beta = \gamma = \frac{1}{4}$.

Conclusion

We have

- defined a model for second order differences of cyclic data
- derived a first and second order TV type functional
- extended the model to inpainting
- employed CPPA in order to minimize the functional
- simultaneous inpainting and denoising

Future work

- combined cyclic and linear vector space data (submitted)
- extension to other manifolds

Literature

[1] M. Bačák. Computing medians and means in Hadamard spaces. *SIAM J. Optim.*, 2014.

[2] R. Bergmann, F. Laus, G. Steidl, and A. Weinmann. Second order differences of cyclic data and applications in variational denoising. *SIAM J. Imaging Sci.*, 2014.

[3] R. Bergmann and A. Weinmann. *A second order TV-type approach for inpainting and denoising higher dimensional combined cyclic and vector space data*. *Preprint, ArXiv, submitted*, 2015.

[4] E. Strekalovskiy and D. Cremers. Total cyclic variation and generalizations. *J. Math. Imaging Vis.*, 2013.

[5] A. Weinmann, L. Demaret, and M. Storath. Total variation regularization for manifold-valued data. *SIAM J. Imaging Sci.*, 2014.

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