Technische Universität Kaiserslautern

Second Order Differences of Cyclic Data and Application to Variational Denoising

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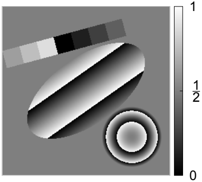
Helmholtz Center Munich

FELIX KLEIN
ZENTRUM FÜR
MATHEMATIK

*joint work with A. Weinmann, G. Steidl, F. Laus

Introduction

What do we want to do?



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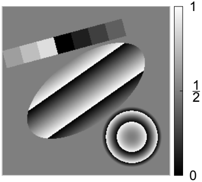

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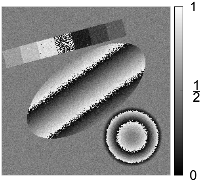

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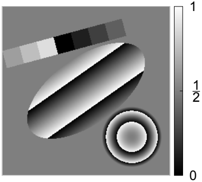

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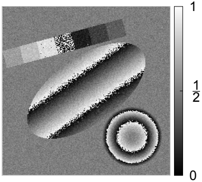

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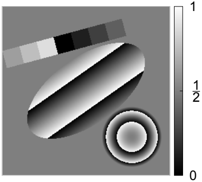

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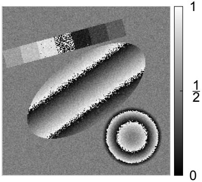

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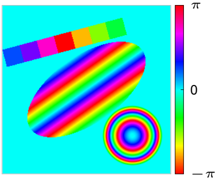

phase valued image f, $f_{i,j} \in \mathbb{S}^1$

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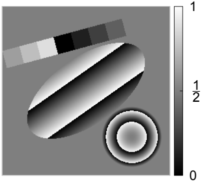

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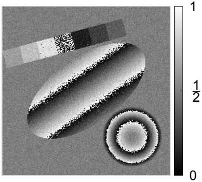

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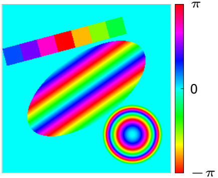

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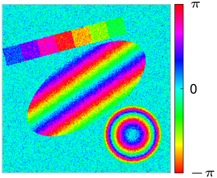

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Goal: reconstruct phase valued image f from noisy data, preserve edges.

Introduction II

[Osher, Rudin, Fatemi, 1992]

- tool: minimizing the Rudin-Osher-Fatemi (ROF) functional

$$\sum_{i,j} (f_{i,j} - x_{i,j})^2 + \lambda \sum_{i,j} |\nabla x_{i,j}|$$

  - ∇ discrete gradient
  - $\sum_{i,j} |\nabla x_{i,j}|$ discrete total variation (TV)
  - regularization parameter $\lambda > 0$

$\Rightarrow$ edge-preserving

- stair caising-effect: reduced by adding higher order derivatives

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Recently [Cremers,Strekalovski, 2012], [Lellmann et al., 2013], [Weinmann et al., 2013]

- TV denoising generalized to Riemannian manifolds
- several algorithms to find the minimizer x^*

Outline

First & Second Order Differences on $\mathbb{R}$

A short reminder.

Let $w = (w_j)_{j=1}^d \in \mathbb{R}^d \setminus \{0\}$ be a weight: $\langle w, 1_d \rangle := \sum_{j=1}^{d} w_j = 0$

The finite difference operator is given by

$$\Delta(x; w) := \langle x, w \rangle, \quad x \in \mathbb{R}^d$$

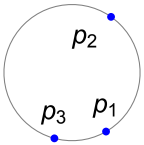
$\Delta(x; w)$ is shift invariant.

Examples

- $b_1 := (-1, 1)$: First order difference $\Delta(x; b_1) = x_2 - x_1$
- $b_2 := (1, -2, 1)$: Second order difference $\Delta(x; b_2) = x_1 - 2x_2 + x_3$
- $b_{1,1} := (-1, 1, 1, -1)$: 'mixed second order difference' $\Delta(x; b_{1,1}) = -x_1 + x_2 + x_3 - x_4$

First & Second Order Difference on $\mathbb{S}^1$

Defined by looking at different situations on $\mathbb{R}$ the points may take.



- $x_i \in [-\pi, \pi) \Leftrightarrow p_i := (\cos x_i, \sin x_i)$
- Idea: unwrap the circle onto any tangential line

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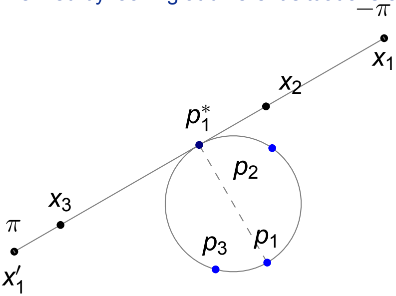

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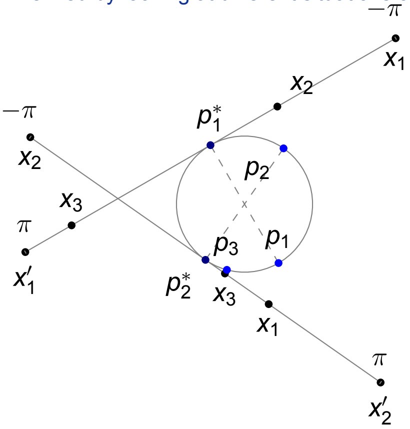

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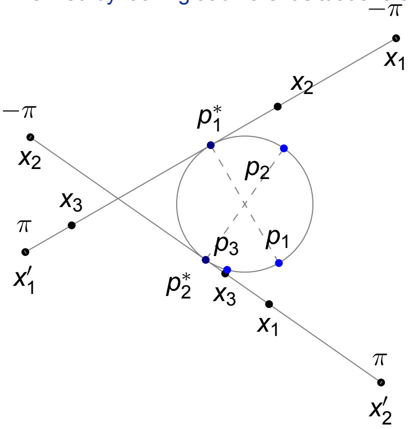

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- Absolute cyclic differences w.r.t w:

$$d(x; w) := \min_{\alpha \in \mathbb{R}} \left| \Delta\left([x + \alpha 1_d]_{2\pi}; w\right) \right|$$

- $[x]_{2\pi}$: element-wise mod 2π except $x_i = (2k+1)\pi$: Δ with $\pm\pi$
- shift invariant

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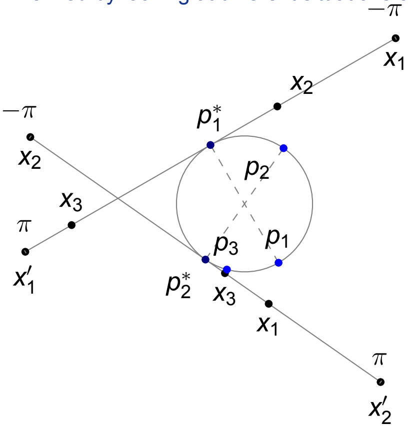

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- b_1: arc length distance $d(x; b_1) = d_1(x_1, x_2)$
- b_2: $d(x; b_2) = d_2(x_1, x_2, x_3) = \left| \left(\Delta(x; b_2)\right)_{2\pi} \right|$ (the same holds for $b_{1,1}$)

Second Order Total Variation on the Circle

Transfer the ROF functional to the circle.

Let $f = (f_i)_{i=1}^N$ be given data on $\mathbb{S}^1$, $\alpha, \beta \geq 0$.
We are interested in the minimizers x^* of

$$J(x) := F(x; f) + \alpha\,\mathrm{TV}_1(x) + \beta\,\mathrm{TV}_2(x),$$

where

- data fidelity term $F(x; f) = \frac{1}{2}\sum_{i=1}^{N} d_1(f_i, x_i)^2$
- first order differences $\mathrm{TV}_1(x) = \sum_{i=1}^{N-1} d_1(x_i, x_{i+1})$
- second order differences $\mathrm{TV}_2(x) = \sum_{i=2}^{N-1} d_2(x_{i-1}, x_i, x_{i+1})$

Proximal Point Algorithm

For a proper, closed, convex function $\varphi\colon \mathbb{R}^N \to (-\infty, +\infty]$ and $\lambda > 0$ the proximal mapping $\operatorname{prox}_{\lambda\varphi}\colon \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$\operatorname{prox}_{\lambda\varphi}(f) := \operatorname*{arg\,min}_{x\in\mathbb{R}^N} \frac{1}{2}\|f - x\|_2^2 + \lambda\varphi(x),$$

- trade-off: minimizing φ vs. "staying near" f
- λ: weight or trade-off parameter
- fixpoints of $\operatorname{prox}_{\lambda\varphi}$: minima of φ.
- often: closed form of $\operatorname{prox}_{\lambda\varphi}$ known.

Proximal Point Algorithm (PPA)

[Moreau, 1965; Rockafellar, 1976]

$$x^{(k+1)} = \operatorname{prox}_{\lambda\varphi}(x^{(k)}), \quad k \in \mathbb{N}$$

Cyclic Proximal Point Algorithm

Split into smaller proximal mappings and iterate.

- $\varphi = \sum_{i=1}^{c} \varphi_i,$ c is called the cycle length,
- proximal mappings of summands φ_i "easier"

$\Rightarrow$ iteratively apply "small" proximal mappings $\operatorname{prox}_{\lambda\varphi_i}$

Cyclic Proximal Point Algorithm (CPPA)

$$x^{(k+\frac{i+1}{c})} = \operatorname{prox}_{\lambda_k\varphi_i}(x^{(k+\frac{i}{c})}), \quad i = 0, \dots, c-1, k \in \mathbb{N}.$$

Lemma (Convergence of the CPPA on $\mathbb{R}$ [Bertsekas, 2011])

Let φ have a minimizer x^* and $\{\lambda_k\}_k$ be a sequence, such that

- $\sum_{k=1}^{\infty} \lambda_k = \infty$
- $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$

Then the CPPA converges to a minimizer.

Proximal Mapping I

for each data fidelity term of data on $\mathbb{S}^1$.

- data fidelity term $\varphi(x) = d_1(f,x)^2, f \in [-\pi,\pi)$
- $\operatorname{prox}_{\lambda d_1(f,\cdot)^2}(g) = \arg\min_x \frac{1}{2} d_1(g,x)^2 + \lambda d_1(f,x)^2$
- idea again: "near g" vs. minimizing $d_1(f,x)^2$

Theorem (B., Laus, Steidl, Weinmann)

The unique minimizer x^* of $\operatorname{prox}_{\lambda d_1(f,\cdot)^2}(g)$ is

$$x^* = \left(\frac{g+\lambda f}{1+\lambda} + \frac{\lambda}{1+\lambda} 2\pi v\right)_{2\pi}, \quad v = \begin{cases} 0 & \text{for } |g-f| \leq \pi, \\ \operatorname{sgn}(g-f) & \text{for } |g-f| > \pi \end{cases}$$

Sketch of proof

- first term is the minimizer on $\mathbb{R}$
- second term the minimial value, taking $g + 2\pi k$, $f + 2\pi l$ into account

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Proximal Mapping II

for the finite difference terms on $\mathbb{S}^1$.

- finite difference term $\varphi(x) = d(x; w), w \in \{b_1, b_2, b_{1,1}\}$
- x, g same length as w
- $\operatorname{prox}_{\lambda d(\cdot;w)}(g) = \arg\min_x d_1(g,x)^2 + \lambda d(x;w)$

Theorem (B., Laus, Steidl, Weinmann)

Set $s := \operatorname{sgn}(\langle g, w\rangle)_{2\pi}$ *and* $m := \min\left\{\lambda, \frac{|(\langle g,w\rangle)_{2\pi}|}{\|w\|_2^2}\right\}$.

1. *If* $|(\langle g, w\rangle)_{2\pi}| < \pi$, *the unique minimizer is given by*

$$x^* = (g - s\,m\,w)_{2\pi}$$

2. *If* $|(\langle g, w\rangle)_{2\pi}| = \pi$, *the two minimizers are*

$$x^* = (g \mp s\,m\,w)_{2\pi}$$

Idea of the proof: Minimizing over “possible constellations” on $\mathbb{R}$.

CPPA with Second Order TV for 1D Data on $\mathbb{S}^1$

How to split the higher order TV functional J?

- $F(x;f) = \frac{1}{2}\sum_{i=1}^{N} d_1(f_i,x_i)^2 =: J_1(x)$

proximal mapping I (applied element-wise)

- first order differences

$$\alpha\,\mathrm{TV}_1(x) = \alpha\sum_{i=1}^{N-1} d_1(x_i,x_{i+1})$$

- second order differences

$$\beta\,\mathrm{TV}_2(x) = \beta\sum_{i=2}^{N-1} d_2(x_{i-1},x_i,x_{i+1})$$

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  inner sum: distinct data $\Rightarrow$ proximal mapping II with $w = b_1$

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inner sum: distinct data $\Rightarrow$ proximal mapping II with $w = b_2$

CPPA with Second Order TV for 1D Data on $\mathbb{S}^1$

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$\Rightarrow$ $J(x) = \sum_{l=1}^{6} J_l(x)$, i.e., cycle length $c = 6$

Algorithm for CPP on $\mathbb{S}^1$

Input non-negative parameters $\lambda_0 > 0$ and α, β
data $f \in [-\pi, \pi)^N$

CPPA(α, β, λ_0, f)

Initialize $x^{(0)} \leftarrow f$, $k \leftarrow 0$

Initialize the cycle length $c \leftarrow 6$

Repeat

For l from 1 to c

$x^{(k-1+\frac{l}{c})} \leftarrow \operatorname{prox}_{\lambda_k J_l}(x^{(k-1+\frac{l-1}{c})})$

$k \leftarrow k+1$

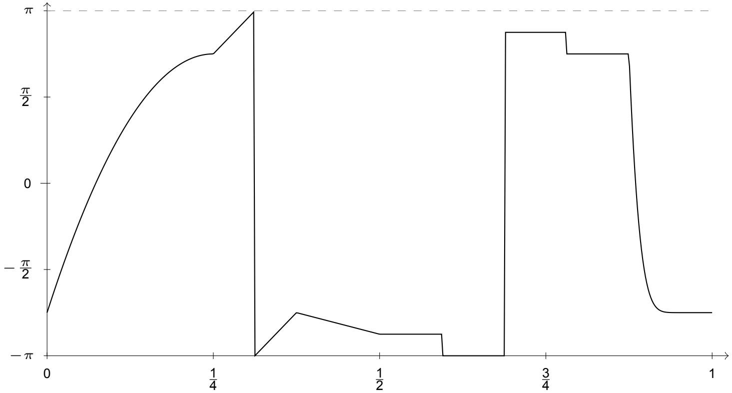
$\lambda_k \leftarrow \frac{\lambda_0}{k}$

Until a convergence criterion are reached

Return $x^{(k)}$

Example

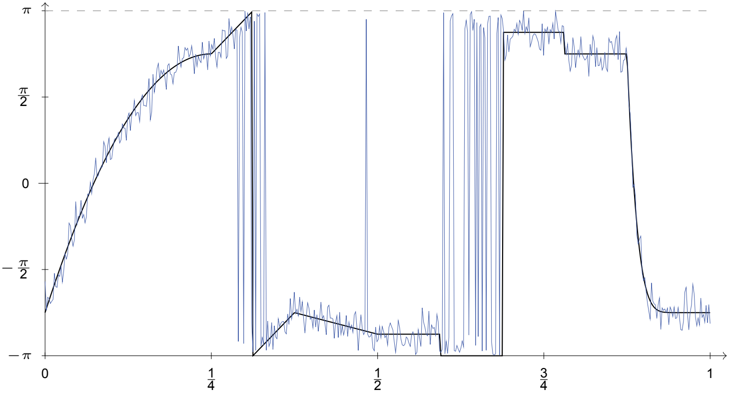
Denoising a 1D phase valued signal.



- function $f\colon [0,1] \to \mathbb{S}^1$ sampled to obtain data $f_{\text{o}} = (f_{\text{o},i})_{i=1}^{500}$
- jumps $> \pi$ at $\frac{5}{16}$ and $\frac{11}{16}$ are due to the representation system

Example

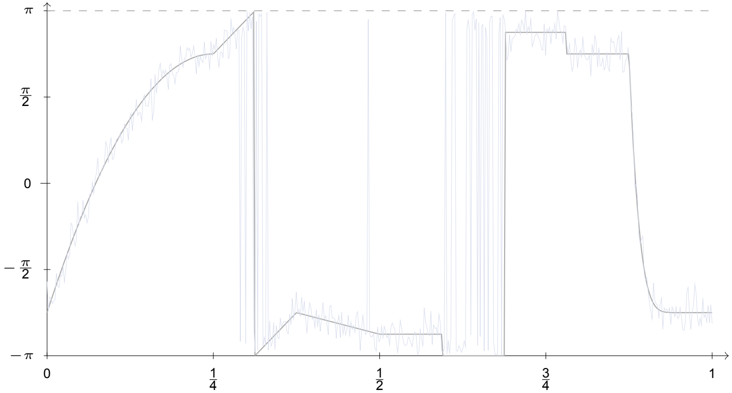
Denoising a 1D phase valued signal.



- function $f\colon [0,1] \to \mathbb{S}^1$ sampled to obtain data $f_{\mathrm{o}} = (f_{\mathrm{o},i})_{i=1}^{500}$
- adding wrapped Gaussian noise, $\sigma = 0.2$
- noisy data $f_{\mathrm{n}} = (f_0 + \eta)_{2\pi}$

Example

Denoising a 1D phase valued signal.



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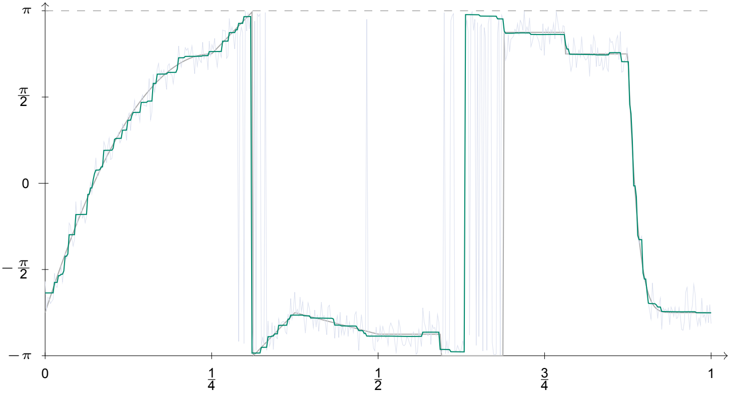

- comparison of f_o & f_n with f_1
- denoising: just TV_1: $\alpha = \frac{3}{4}$, $\beta = 0$
- but: stair casing

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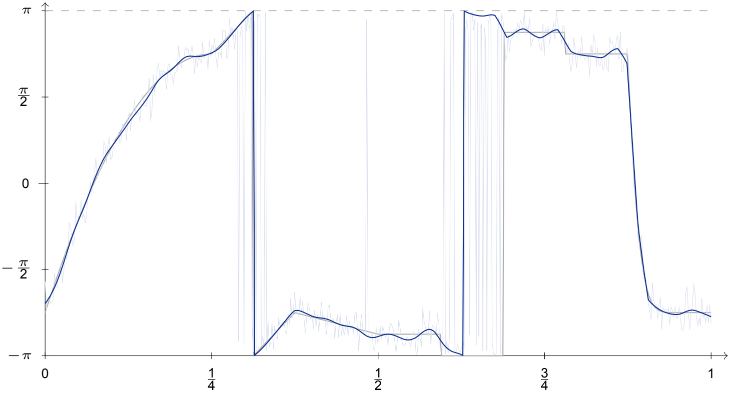

- comparison of f_o & f_n with f_2
- denoising: just TV_2: $\alpha = 0$, $\beta = \frac{3}{2}$
- but: no plateaus

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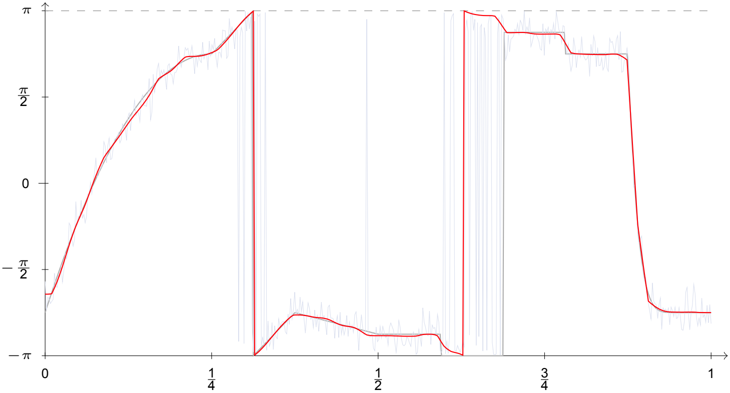
Denoising a 1D phase valued signal.



- comparison of f_o & f_n with f_3
- denoising: TV$_1$ & TV$_2$: $\alpha = \frac{1}{2}$, $\beta = 1$
- smallest mean squared error

CPPA with Second Order TV for 2D data on $\mathbb{S}^1$

Splitting the functional J for an $N \times M$ pixel image using mainly things we already know.

data $f := (f_{i,j})_{i,j=1}^{N,M} \in [-\pi, \pi)^{N\times M}$ and $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, γ

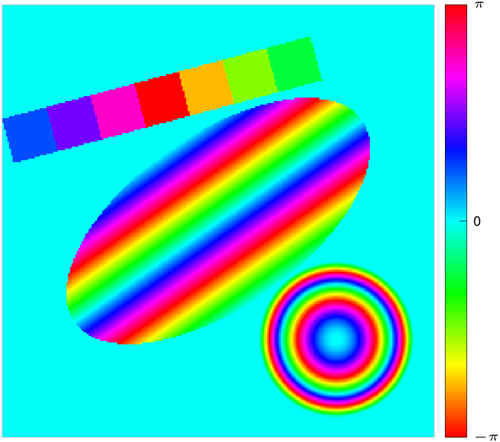
- $F(x; f)$ element-wise distance as before
- $$\alpha\, \mathrm{TV}_1(x) := \alpha_1 \sum_{i,j=1}^{N-1,M} d_1(x_{i,j}, x_{i+1,j}) + \alpha_2 \sum_{i,j=1}^{N,M-1} d_1(x_{i,j}, x_{i,j+1})$$
- $$\beta\, \mathrm{TV}_2^{\mathrm{hv}}(x) := \beta_1 \sum_{i=1,j=2}^{N-1,M} d_2(x_{i-1,j}, x_{i,j}, x_{i+1,j}) + \beta_2 \sum_{i=2,j=1}^{N,M-1} d_2(x_{i,j-1}, x_{i,j}, x_{i,j+1})$$
- $$\gamma\, \mathrm{TV}_2^{\mathrm{d}}(x) := \gamma \sum_{i,j=1}^{N-1,M-1} d_{1,1}(x_{i,j}, x_{i+1,j}, x_{i,j+1}, x_{i+1,j+1})$$

$\Rightarrow$ minimizing $J(x) := F(x; f) + \alpha\, \mathrm{TV}_1(x) + \beta\, \mathrm{TV}_2^{\mathrm{hv}}(x) + \gamma\, \mathrm{TV}_2^{\mathrm{d}}(x)$

- data term, 2×2 TV_1 terms, 2×3 $\mathrm{TV}_2^{\mathrm{hv}}$ terms, 4 $\mathrm{TV}_2^{\mathrm{d}}$ terms $\Rightarrow c = 15$

Example

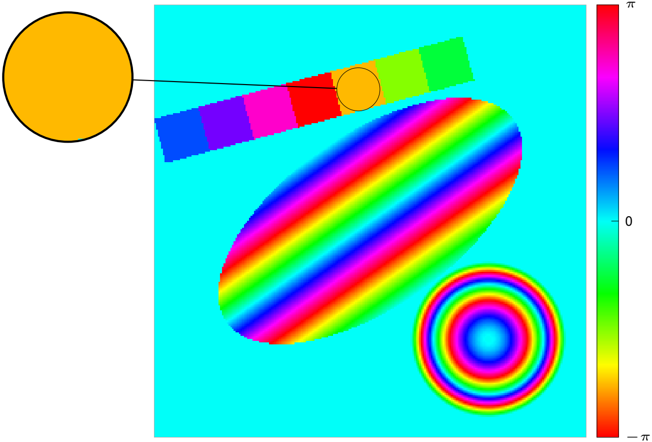
Denoising artificial phase valued data.



original data f_o, 256×256 pixel image

Example

Denoising artificial phase valued data.



original data f_o, 256×256 pixel image

Example

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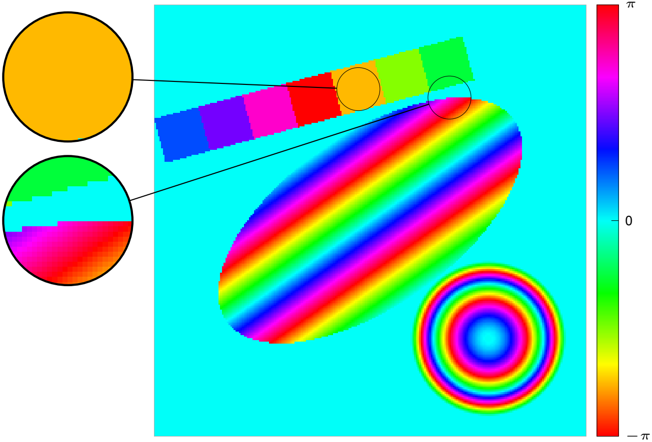

original data f_o, 256×256 pixel image

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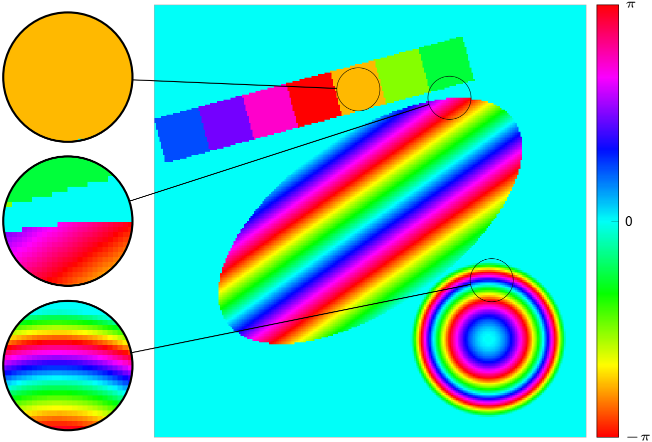
Denoising artificial phase valued data.



original data f_o, 256×256 pixel image

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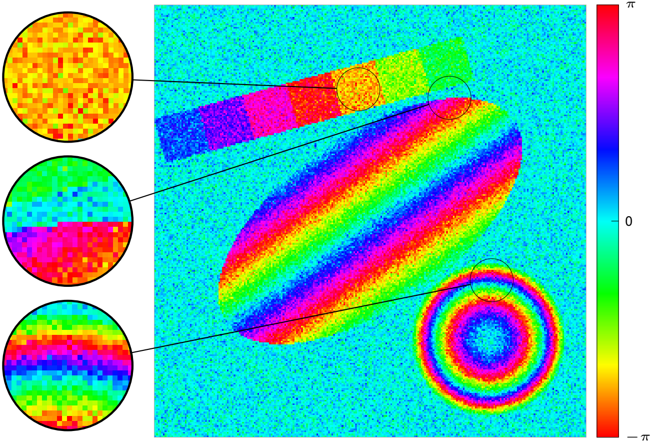
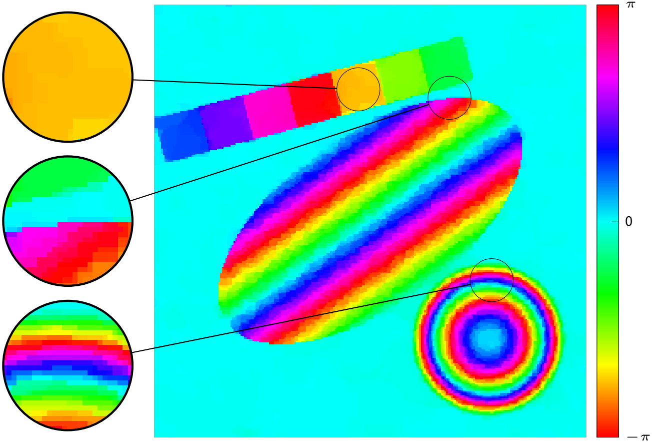


noisy data f_{n}, $\sigma = 0.3$

Example

Denoising artificial phase valued data.

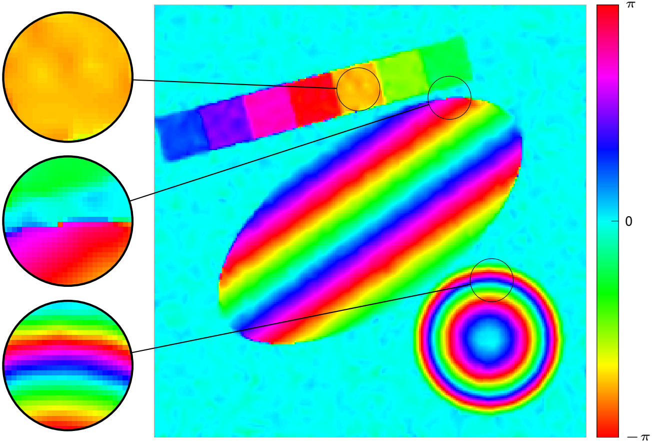


denoising f_n: f_1 with just TV_1

$\alpha_1 = \frac{3}{8}, \alpha_2 = \frac{1}{4}, \beta_1 = \beta_2 = \gamma = 0$: stair casing

Example

Denoising artificial phase valued data.

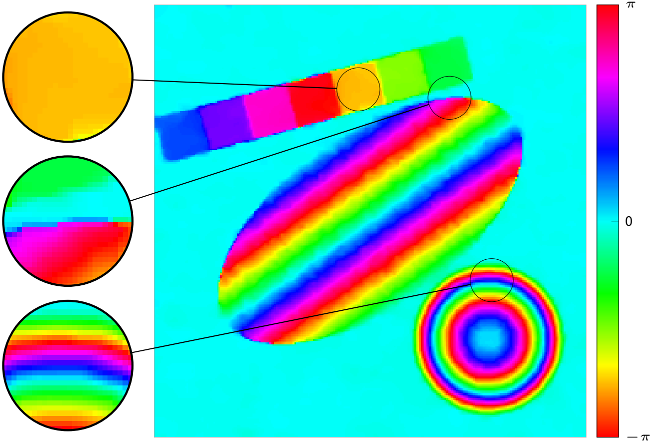


denoising f_n: f_2 with just TV_2

$\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = \gamma = \frac{1}{8}$: no plateaus

Example

Denoising artificial phase valued data.



denoising f_n: f_3 with TV_1 & TV_2

$\alpha_1 = \frac{1}{4}$, $\alpha_2 = \beta_1 = \beta_2 = \frac{1}{8}, \gamma = 0$: smallest mean squared error

Convergence of CPPA on $\mathbb{S}^1$

Comparison to $\mathbb{R}$ and challenges.

On $\mathbb{R}$ and Hadamard spaces [Bačák, 2013]
(e.g. Riemannian manifold, non-pos. curv.,simply connected)

- $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\sum_{k=0}^{\infty} \lambda_k^2 < \infty$

$\Rightarrow$ CPPA on $\mathbb{R}$ converges (weakly) to a global minimizer

- proof uses i.a. convexity of J_i
- How to define convexity on $\mathbb{S}^1$?

Example

For $x_0 \in \mathbb{S}^1$ take

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) := d_1\bigl(x_0, (x)_{2\pi}\bigr).$$

Then f is not convex.

Convergence of CPPA on $\mathbb{S}^1$

With restriction on data f and λ_0.

Theorem (B., Laus, Steidl, Weinmann)

Let $x^{(0)} = f$. And for an $\varepsilon > 0$

- $\mathrm{TV}_1(f) + \mathrm{TV}_2^{hv}(f) + \mathrm{TV}_2^{d}(f) \leq \frac{\varepsilon^2}{\max\{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma\}}$
- $\max_{i,j} \max\{d_1(f_{i,j}, f_{i,j+1}), d_1(f_{i,j}, f_{i+1,j})\} \leq \frac{\pi}{8}$
- ε, λ_0 *and* $\|\lambda\|_2^2$ *are "small enough"*

Then the CPPA on $\mathbb{S}^1$ converges to a minimizer x^*.

Ideas of the proof:

- "control" $\sum d_1(x^{(k+\frac{i}{c})}, f)$ and from $x_{i,j}^{(k+\frac{i}{c})}$ to its 4-neighborhood
- assure, that this still holds after applying the proximal mappings

$\Rightarrow$ all involved J_i have a convex analogue on $\mathbb{R}$

Denoising of InSAR Data

Measuring earth elevation from radar data.

Synthetic Aperture Radar

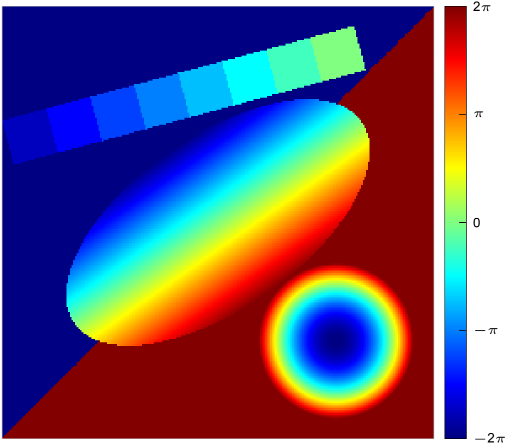
- emit radar & use motion of antenna (i.e. speed of airplane)
- record amplitude and phase of an backscattered signal
- amplitude: reflectivity of the surface
- phase: both elevation and reflection properties
- ! phase of one SAR data rather arbitrary
- record certain area $\Rightarrow$ SAR image

Interferometry

- take two SAR images with different (but known) angles or locations
- $\Rightarrow$ phase difference: principal or wrapped phase
- encodes elevation, but is noisy

Artificial Example

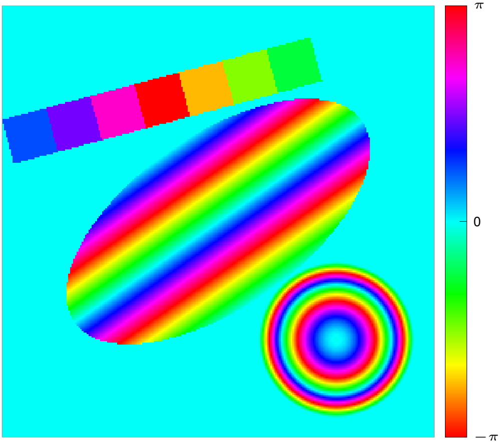
Illustrating the effect of wrapped phase & noise



elevation profile

Artificial Example

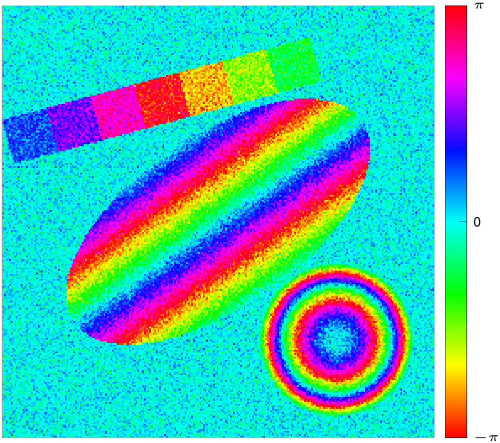
Illustrating the effect of wrapped phase & noise



wrapped phase

Artificial Example

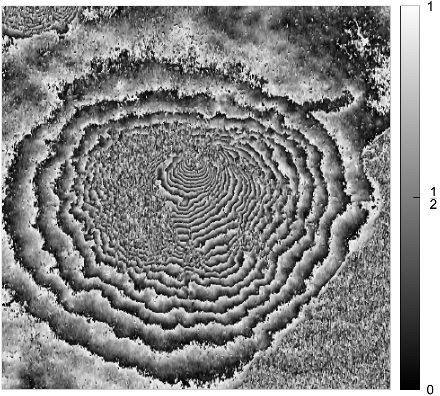
Illustrating the effect of wrapped phase & noise



added noise

Mount Vesuvius

The following image is InSAR data from Mount Vesuvius, Italy.[1]



original data, 432×426 pixel

[1] https://earth.esa.int/workshops/ers97/program-details/speeches/rocca-et-al/

Mount Vesuvius

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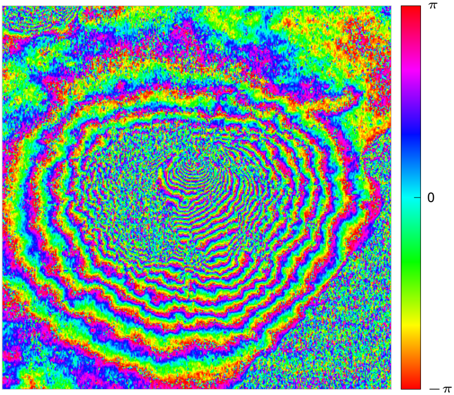

adapted just the coloring

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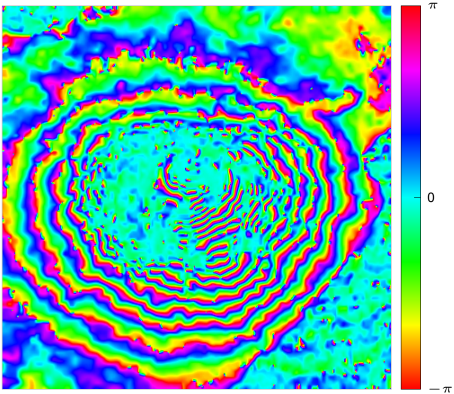

denoised: $\alpha_1 = \alpha_2 = \frac{1}{4}, \beta_1 = \beta_2 = \gamma = \frac{3}{4}$

[1] https://earth.esa.int/workshops/ers97/program-details/speeches/rocca-et-al/

Conclusion

We derived for $\mathbb{S}^1$-valued 1D & 2D data f

- higher order differences
- proximal mappings for first and second order differences
- higher order TV functional J
- an efficient CPPA to minimize J
- convergence
- application: InSAR data denoising $\Rightarrow$ goal: unwrapping

Future work

- loosen contraints of convergence
- further applications of TV (impainting,...)

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Thank you for your attention.