Multivariate Periodic Function Spaces*

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Sparse Representations and Efficient Sensing of Data

* Joint work with J. Prestin

Contents

Definitions

On $\mathbb{T}^d \cong [0, 2\pi)^d$, the d-dimensional torus, we define the inner product

$$\langle f, g\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\mathbf{x})\overline{g(\mathbf{x})}\, d\mathbf{x}, \quad f, g : \mathbb{T}^d \to \mathbb{C}$$

and norm $\|f\|^2 = \langle f, f\rangle$
$L^2(\mathbb{T}^d) := \{f : \|f\| < \infty\}$ is a Hilbert space

Analogously:

$$\langle \mathbf{a}, \mathbf{b}\rangle_{l^2} = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}}\overline{b_{\mathbf{k}}}, \quad \mathbf{a}, \mathbf{b} \in l^2(\mathbb{Z}^d)$$

and norm $\|\mathbf{a}\|_{l^2}$.

The pattern and the generating group

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be invertible. The lattice $\Lambda(\mathbf{M}) := \mathbf{M}^{-1}\mathbb{Z}^d$ is 1-periodic.
Let the *pattern* $\mathcal{P}(\mathbf{M})$ denote a full collection of coset representations for $+ \bmod \mathbf{I}$, e.g.

$$\mathcal{P}(\mathbf{M}) := \Lambda(\mathbf{M}) \cap [0,1)^d$$

i.e. every $\mathbf{x} \in \Lambda(\mathbf{M})$ can be written as

$$\mathbf{x} = \mathbf{y} + \mathbf{z}, \quad \mathbf{y} \in \mathcal{P}(\mathbf{M}), \mathbf{z} \in \mathbb{Z}^d. \tag{1}$$

We define the *generating group* $\mathcal{G}(\mathbf{M}) := \mathbf{M}\mathcal{P}(\mathbf{M})$.
$\mathcal{G}(\mathbf{M})$ is a full collection of coset representations for $+ \bmod \mathbf{M}$, i.e.

$$\mathbf{k} = \mathbf{h} + \mathbf{M}\mathbf{z}, \quad \mathbf{h} \in \mathcal{G}(\mathbf{M}), \mathbf{z} \in \mathbb{Z}^d \tag{2}$$

Smith normal form and cycles in $\mathcal{P}(\mathbf{M})$

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be invertible. The Smith normal form is a decomposition

$$\mathbf{M} = \mathbf{QER},$$

$|\det \mathbf{R}| = |\det \mathbf{Q}| = 1$ and $\mathbf{E} = \operatorname{diag}(\varepsilon_1, \ldots \varepsilon_d)$
with the elementary divisors $\varepsilon_{j-1}|\varepsilon_j$, $j = 2, \ldots, d$.

$\mathbf{Q}$ and $\mathbf{R}$ just perform a change of basis $\Rightarrow \mathcal{P}(\mathbf{M}) \cong \mathcal{P}(\mathbf{E})$
$\mathcal{P}(\mathbf{E}) = \mathcal{C}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{C}_{\varepsilon_d}$ is a direct sum of cycles, $\mathcal{C}_{\varepsilon_j} = \frac{1}{\varepsilon_j}\mathbf{e}_j\{0, \ldots, \varepsilon_j - 1\}$.

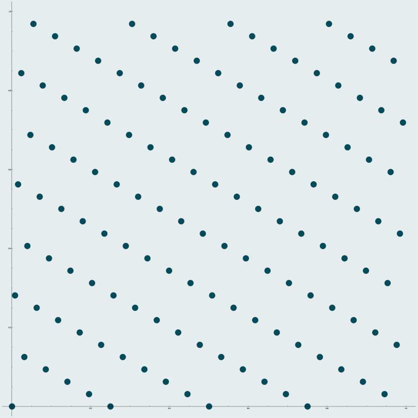
$\Rightarrow \#\mathcal{P}(\mathbf{M}) = \#\mathcal{G}(\mathbf{M}) = \#\mathcal{P}(\mathbf{M}^T) = |\det \mathbf{M}| = \varepsilon_1 \cdot \ldots \cdot \varepsilon_d$

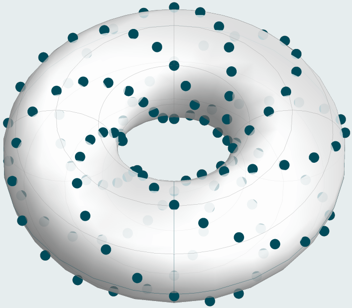
Example

of a Smith normal Form

$$\mathcal{P}\left(\begin{pmatrix} -24 & -10 \\ 20 & 3 \end{pmatrix}\right) = \mathcal{P}\left(\begin{pmatrix} 4 & 7 \\ 0 & 32 \end{pmatrix}\right), \text{ but}$$

$$\varepsilon_1 = 1, \varepsilon_2 = 128$$





Translation invariant subspaces

For $f \in L^2(\mathbb{T}^d)$ and $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ the translation operator is given by

$$T(\mathbf{y})f := f(\circ - 2\pi\mathbf{y}).$$

A linear subspace $V \subset L^2(\mathbb{T}^d)$ is called $\mathbf{M}$-invariant if

$$f \in V \Rightarrow T(\mathbf{y})f \in V \quad \text{for all } \mathbf{y} \in \mathcal{P}(\mathbf{M}).$$

Lemma

For any $f \in L^2(\mathbb{T}^d)$ the span of translates (w.r.t. $\mathbf{M}$), i.e.
$V_{\mathbf{M}}^f := \operatorname{span}\{T(\mathbf{y})f : \mathbf{y} \in \mathcal{P}(\mathbf{M})\}$*, is $\mathbf{M}$–invariant*

Proof.

Translation performs an index shift $(\operatorname{mod} \mathbf{I})$ on $g = \sum_{\mathbf{y} \in \mathcal{P}(\mathbf{M})} a_{\mathbf{y}} T(\mathbf{y}) f \in V_{\mathbf{M}}^f$. □

Fourier series

For all $f \in L^2(\mathbb{T}^d)$ it holds

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(f)\,\mathrm{e}^{\mathrm{i}\mathbf{k}^T\circ}, \text{ where } c_{\mathbf{k}}(f) = \langle f, \mathrm{e}^{\mathrm{i}\mathbf{k}^T\circ}\rangle, \quad \mathbf{c} = (c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d).$$

Parseval equation for $f, g \in L^2(\mathbb{T}^d)$:

$$\langle f, g\rangle = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(f)\overline{c_{\mathbf{k}}(g)}$$

Lemma

The Fourier coefficients of $T(\mathbf{y})f$ are

$$c_{\mathbf{k}}(T(\mathbf{y})f) = c_{\mathbf{k}}(f(\circ - 2\pi\mathbf{y})) = \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^T\mathbf{y}}\,c_{\mathbf{k}}(f)$$

(Fast) Fourier transform on $\mathcal{P}(\mathbf{M})$

Let $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be invertible. The Fourier matrix $\mathcal{F}(\mathbf{M})$ is defined by

$$\mathcal{F}(\mathbf{M}) := \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^T\mathbf{y}}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^T),\mathbf{y}\in\mathcal{P}(\mathbf{M})} \in \mathbb{C}^{m\times m}, \quad m = |\det\mathbf{M}|.$$

Performs a DFT for any $\mathbf{a} = (a_{\mathbf{y}})_{\mathbf{y}\in\mathcal{P}(\mathbf{M})}$ by $\hat{\mathbf{a}} = (\hat{a}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M})} = \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a}$

Permutations on rows and columns together with elementary divisors

$$\mathcal{F}(\mathbf{M}) = \mathbf{P}_{\mathbf{h}}\mathcal{F}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{F}_{\varepsilon_d}\mathbf{P}_{\mathbf{y}}, \quad \mathcal{F}_{\varepsilon} = \left(\mathrm{e}^{-2\pi\mathrm{i}h\varepsilon^{-1}g}\right)_{g,h=0}^{\varepsilon-1},$$

where $\mathbf{P}_{\mathbf{h}}, \mathbf{P}_{\mathbf{y}}$ permute the elements of $\mathcal{G}(\mathbf{M}^T)$ and $\mathcal{P}(\mathbf{M})$ respectively, hence $\mathcal{F}(\mathbf{M})\overline{\mathcal{F}(\mathbf{M})}^T = \mathbf{I} \in \mathbb{C}^{m\times m}$.

This is used to obtain an implementation of the FFT ($O(m\log m)$).

Characterizing subspaces

Lemma

Let $\mathbf{M} = \mathbf{J}\mathbf{N}$ *be a decomposition of a regular matrix* $\mathbf{M},\ \mathbf{N}, \mathbf{J} \in \mathbb{Z}^{d\times d}$. *Then*

$$\mathcal{P}(\mathbf{N}) \subset \mathcal{P}(\mathbf{M})$$

Theorem

$g \in V_{\mathbf{M}}^{f}$ *holds iff there exists* $\mathbf{a} = (a_{\mathbf{y}})_{\mathbf{y}\in\mathcal{P}(\mathbf{M})}$ *with DFT* $\hat{\mathbf{a}} = (\hat{a}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^T)} = \sqrt{m}\mathcal{F}(\mathbf{M})\mathbf{a}$, *such that*

$$c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(g) = \hat{a}_{\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}, \quad \mathbf{h} \in \mathcal{G}(\mathbf{M}^T),\ \mathbf{z} \in \mathbb{Z}^d \tag{3}$$

Hence $V_{\mathbf{N}}^{g} \subset V_{\mathbf{M}}^{f}$

Characterizing subspaces

Proof.

$g \in V_{\mathbf{M}}^f$ iff g can be written as

$$g = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{\mathbf{y}} T(\mathbf{y}) f$$

$$\Leftrightarrow c_{\mathbf{k}}(g) = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{\mathbf{y}} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^T\mathbf{y}} c_{\mathbf{k}}(f), \quad \mathbf{k}\in\mathbb{Z}^d$$

rewriting $\mathbf{k} = \mathbf{h} + \mathbf{M}^T\mathbf{z}$, $\mathbf{h} \in \mathcal{G}(\mathbf{M}^T)$, $\mathbf{z}\in\mathbb{Z}^d$ and with $\mathrm{e}^{-2\pi\mathrm{i}\mathbf{z}^T\mathbf{M}\mathbf{y}} = 1$

$$c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(g) = \sum_{\mathbf{y}\in\mathcal{P}(\mathbf{M})} a_{\mathbf{y}} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^T\mathbf{y}} c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f) = \hat{a}_{\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f)$$

□

Gram matrix of the translates

Let $f \in L^2(\mathbb{T}^d)$, $\mathbf{M} \in \mathbb{Z}^{d\times d}$ be invertible and denote $\mathbf{f} = (T(\mathbf{y})f)_{\mathbf{y}\in\mathcal{P}(\mathbf{M})}$. The *Gram matrix* is defined by

$$\mathbf{G}(\mathbf{f}) := (\langle T(\mathbf{y})f, T(\mathbf{x})f\rangle)_{\mathbf{x},\mathbf{y}\in\mathcal{P}(\mathbf{M})} = (\langle f, T(\mathbf{x}-\mathbf{y})f\rangle)_{\mathbf{x},\mathbf{y}\in\mathcal{P}(\mathbf{M})}$$

Hence $\mathbf{G}(\mathbf{f})$ is circular.

Theorem

The Gram Matrix fulfills

$$\mathbf{G}(\mathbf{f}) = \mathcal{F}(\mathbf{M})\,\mathrm{diag}\left(m\sum_{\mathbf{z}\in\mathbb{Z}^d}|c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f)|^2\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^T)}\overline{\mathcal{F}(\mathbf{M})}^T$$

Proof of diagonalization of the Gram matrix

Proof.

$$
\begin{aligned}
\bigl(\langle f, T(\mathbf{x}-\mathbf{y})f\rangle\bigr)_{\mathbf{x},\mathbf{y}\in\mathcal{P}(\mathbf{M})} &= \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(f)\overline{\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^T(\mathbf{x}-\mathbf{y})}\, c_{\mathbf{k}}(f)}\right)_{\mathbf{x},\mathbf{y}\in\mathcal{P}(\mathbf{M})}\\
&= \left(\sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^T)}\sum_{\mathbf{z}\in\mathbb{Z}^d} \mathrm{e}^{-2\pi\mathrm{i}(\mathbf{h}+\mathbf{M}^T\mathbf{z})^T(\mathbf{y}-\mathbf{x})}\, |c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f)|^2\right)_{\mathbf{x},\mathbf{y}\in\mathcal{P}(\mathbf{M})}\\
&= \left(\sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^T)} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^T(\mathbf{y}-\mathbf{x})}\sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f)|^2\right)_{\mathbf{x},\mathbf{y}\in\mathcal{P}(\mathbf{M})}\\
&= \frac{m}{\sqrt{m}}\left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^T\mathbf{y}}\sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f)|^2\right)_{\mathbf{y}\in\mathcal{P}(\mathbf{M}),\mathbf{h}\in\mathcal{G}(\mathbf{M}^T)}\\
&\quad\times\frac{1}{\sqrt{m}}\left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{h}^T\mathbf{x}}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^T),\mathbf{x}\in\mathcal{P}(\mathbf{M})}
\end{aligned}
$$

□

Orthonormal bases for $V_{\mathbf{M}}^{f}$

Lemma

The set $\{T(\mathbf{y})f : \mathbf{y} \in \mathcal{P}(\mathbf{M})\}$ *is linearly independent iff*

$$\forall \mathbf{h} \in \mathcal{G}(\mathbf{M}^T) : \sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f)|^2 > 0$$

Lemma

$\{T(\mathbf{y})f : \mathbf{y} \in \mathcal{P}(\mathbf{M})\}$ *are orthonormal iff*

$$\forall \mathbf{h} \in \mathcal{G}(\mathbf{M}^T) \sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^T\mathbf{z}}(f)|^2 = \frac{1}{m}$$

Holds due to $\displaystyle\sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^T)} \frac{1}{m} \mathrm{e}^{-2\pi \mathrm{i} \mathbf{h}^T(\mathbf{y}-\mathbf{x})} = \begin{cases} 1 & \mathbf{x}=\mathbf{y} \\ 0 & \text{else} \end{cases}$

$\Rightarrow$ Orthonormalization of a basis

Orthogonal decomposition

Let

- $\mathbf{M} = \mathbf{J}\mathbf{N}$ invertible and $|\det \mathbf{J}| = 2 \Rightarrow \mathbf{p} \in \mathcal{G}(\mathbf{J}^T)\setminus\{\mathbf{0}\}$ is unique.
- $f \in L^2(\mathbb{T}^d)$ with $\dim V_{\mathbf{M}}^f = m$ (Translates $T(\mathbf{y})f$ are linear independent)
- $g \in V_{\mathbf{M}}^f$ with $\dim V_{\mathbf{N}}^g = n = |\det \mathbf{N}|$, where
  $\hat{\mathbf{a}} = (\hat{a}_{\mathbf{k}})_{\mathbf{k}\in\mathcal{G}(\mathbf{M}^T)}$: $c_{\mathbf{k}+\mathbf{M}^T\mathbf{z}}(g) = \hat{a}_{\mathbf{k}} c_{\mathbf{k}+\mathbf{M}^T\mathbf{z}}(f)$ for all $\mathbf{k} \in \mathcal{G}(\mathbf{M}^T)$, $\mathbf{z} \in \mathbb{Z}^d$

Goal: Decompose

$$V_{\mathbf{M}}^f = V_{\mathbf{N}}^g \oplus V_{\mathbf{N}}^h \Leftrightarrow h \in V_{\mathbf{M}}^f : \langle T(\mathbf{y})g, T(\mathbf{x})h\rangle = 0,\ \forall \mathbf{x},\mathbf{y} \in \mathcal{P}(\mathbf{N}). \tag{4}$$

Theorem

(4) *holds iff* $\exists \sigma_{\mathbf{q}} \in \mathbb{C}\setminus\{\mathbf{0}\}, \mathbf{q} \in \mathcal{G}(\mathbf{M}^T)$ *with* $\sigma_{\mathbf{q}} = -\sigma_{\mathbf{q}+\mathbf{N}^T\mathbf{p}}$ *fulfilling*

$$c_{\mathbf{k}}(h) = \frac{\sigma_{\mathbf{k} \bmod \mathbf{M}^T} \overline{\hat{a}}_{\mathbf{k}+\mathbf{N}^T\mathbf{p} \bmod \mathbf{M}^T}}{\sum\limits_{\mathbf{z}\in\mathbb{Z}^\mathbf{d}} |c_{\mathbf{k}+\mathbf{M}^\mathbf{T}\mathbf{z}}(f)|^2} c_{\mathbf{k}}(f), \quad \mathbf{k} \in \mathbb{Z}^d \tag{5}$$

Orthogonal decomposition

Proof.

$\Rightarrow$) $h \in V_{\mathbf{M}}^{f} \Rightarrow \exists(\hat{b}_{\mathbf{k}})_{\mathbf{k}\in\mathcal{G}(\mathbf{M}^T)} : c_{\mathbf{k}+\mathbf{M}^T\mathbf{z}}(h) = \hat{b}_{\mathbf{k}} c_{\mathbf{k}+\mathbf{M}^T\mathbf{z}}(f), \ \forall \mathbf{k} \in \mathcal{G}(\mathbf{M}^T), \ \mathbf{z} \in \mathbb{Z}^d$.
The vanishing Gram matrix $(\langle T(\mathbf{x})g, T(\mathbf{y})h\rangle)_{\mathbf{x},\mathbf{y}\in\mathcal{P}(\mathbf{N})}$ yields for $\mathbf{k} \in \mathcal{G}(\mathbf{N}^T)$

$$0 = \sum_{\mathbf{z}\in\mathbb{Z}} c_{\mathbf{k}+\mathbf{N}^T\mathbf{z}}(g)\overline{c_{\mathbf{k}+\mathbf{N}^T\mathbf{z}}(h)} = \sum_{\mathbf{p}\in\mathcal{G}(\mathbf{J}^T)} \sum_{\mathbf{z}\in\mathbb{Z}^d} c_{\mathbf{k}+\mathbf{N}^T\mathbf{p}+\mathbf{M}^T\mathbf{z}}(g)\overline{c_{\mathbf{k}+\mathbf{N}^T\mathbf{p}+\mathbf{M}^T\mathbf{z}}(h)}$$

$$= \hat{a}_{\mathbf{k}}\overline{\hat{b}}_{\mathbf{k}} \sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{k}+\mathbf{M}^T\mathbf{z}}(f)|^2 + \hat{a}_{\mathbf{k}+\mathbf{N}^T\mathbf{p}}\overline{\hat{b}}_{\mathbf{k}+\mathbf{N}^T\mathbf{p}} \sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{k}+\mathbf{N}^T\mathbf{p}+\mathbf{M}^T\mathbf{z}}(f)|^2$$

$$\Rightarrow \sigma_{\mathbf{k}} = \frac{\hat{b}_{\mathbf{k}}}{\overline{\hat{a}}_{\mathbf{k}+\mathbf{N}^T\mathbf{p}}} \sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{k}+\mathbf{M}^T\mathbf{z}}(f)|^2 \text{ and } \sigma_{\mathbf{k}+\mathbf{N}^T\mathbf{p}} = \frac{\hat{b}_{\mathbf{k}+\mathbf{N}^T\mathbf{p}}}{\overline{\hat{a}}_{\mathbf{k}}} \sum_{\mathbf{z}\in\mathbb{Z}^d} |c_{\mathbf{k}+\mathbf{N}^T\mathbf{p}+\mathbf{M}^T\mathbf{z}}(f)|^2$$

$\hat{a}_{\mathbf{h}} = \hat{a}_{\mathbf{h}+\mathbf{N}^T\mathbf{p}} = 0$ is impossible due to $\dim V_{\mathbf{N}}^{g} = n$ □

Summary

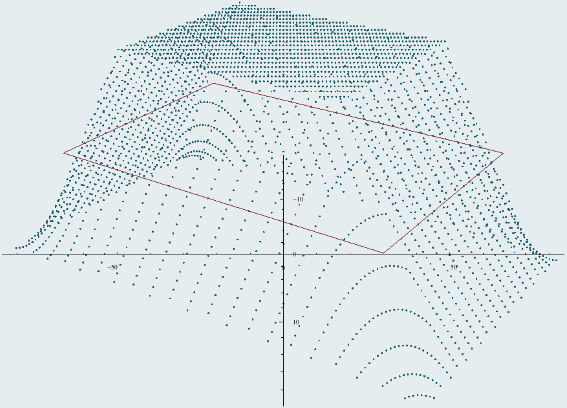
- Smith normal form leads to fast pattern and Fourier algorithms
- basis transforms and decompositions m-dimensional spaces
- decomposition in $V_{\mathbf{M}}^{f}$ into $j = |\det \mathbf{J}|$ subspaces in $\mathcal{O}(m)$.

Perspective

- Classify directions for $\mathcal{P}(\mathbf{M})$ and h or $c_{\mathbf{k}}(h)$
- general wavelet system despite diriclet case
- possible dilation matrices $\mathbf{J}$

Example of a decomposition

Define $f, g, h \in L^2(\mathbb{T}^2)$ as trigonometric Polynomials with a (discrete) Box Splines



spectrum $c_{\mathbf{k}}(f)$

Example of a decomposition

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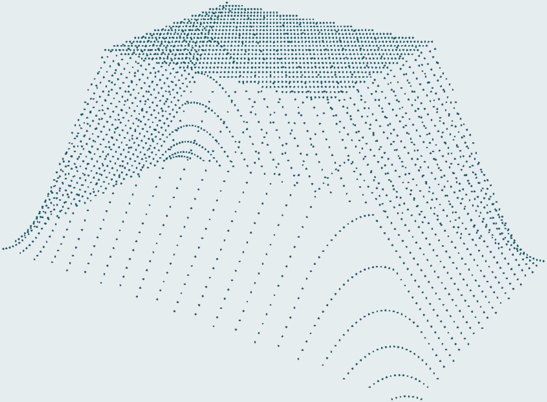

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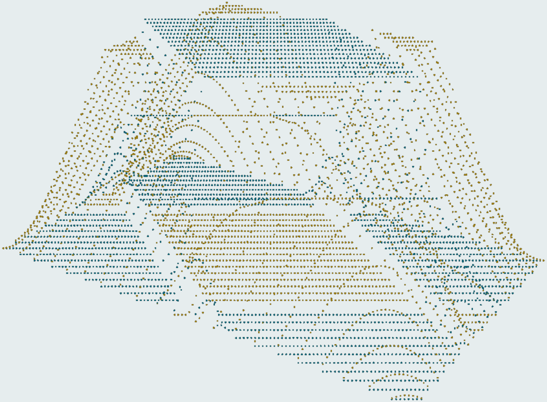

spectrum $c_{\mathbf{k}}(g)$ and $c_{\mathbf{k}}(h)$

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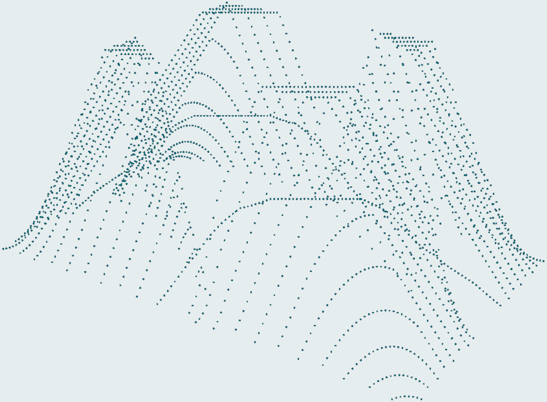

spectrum $c_{\mathbf{k}}(h)$

Example of a decomposition

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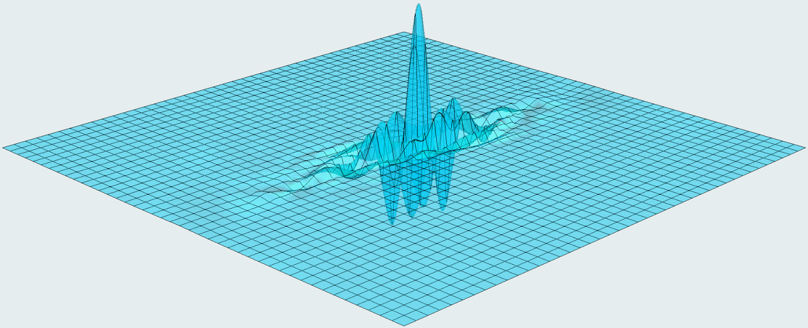

$$h(\mathbf{x}), \quad \mathbf{x} \in [-\pi, \pi)^2$$

Thank you for your attention.

Literature

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